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QUALITATIVE ANALYSIS AND COMPUTATION OF A FLOW OF SURFACE CURVES DRIVEN BY THE GEODESIC CURVATURE

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Abstract. The purpose of this paper is to analytically and numerically investigate a flow of closed curves on a given graph surface driven by the geodesic curvature. We show how such a flow can be reduced to a flow of vertically projected planar curves governed by a solution of a fully nonlinear system of parabolic differential equations. We furthermore analyze closed stationary surface curves and present sufficient conditions for their dynamic stability. Various computational examples of evolution of surface curves driven by the geodesic curvature are presented in this paper.

Keywords: Geodesic curvature, flow of surface curves, linearized stability, closed geodesic curve, curves with prescribed mean curvature

AMS classification: 35K65, 35B35, 35K55, 53A10, 53C44

1. Introduction. The main goal of this paper is to investigate a flow of curves on a given two dimensional surface driven by the geodesic curvature. The normal velocity $\mathcal{V}$ of the evolving family of surface curves $\mathcal{G}_t, t \geq 0,$ is proportional to the geodesic curvature $\mathcal{K}_g$ of $\mathcal{G}_t$, i.e.

$$\mathcal{V} = \delta \mathcal{K}_g \tag{1.1}$$

where $\delta = \delta(X, \vec{\mathcal{N}}) > 0$ is a smooth positive coefficient describing anisotropy depending on the position X and the orientation of the unit inward normal vector $\vec{\mathcal{N}}$ to the curve on a surface.

The idea how to analyze and compute numerically such a flow is based on the so-called direct approach method applied to a flow of vertically projected family of planar curves. Vertical projection of surface curves on a simple surface $\mathcal{M}$ into the plane $\mathbb{R}^2$. It allows for reducing the problem to the analysis of evolution of planar curves $\Gamma_t : S^1 \to \mathbb{R}^2$, $t \geq 0$ driven by the normal velocity v given as a nonlinear function of the position vector x, tangent angle ν and as an affine function of the curvature k of Γ_t, i.e.

$$v = \beta(x, \nu, k) \tag{1.2}$$

where $\beta(x, \nu, k) = a(x, \nu)k + c(x, \nu)$ and $a(x, \nu) > 0, c(x, \nu)$ are smooth coefficients.

In this note we follow the so-called direct approach how to analyze and solve numerically equation (1.2). The direct approach has been first used by Gage and Hamilton [6], and Grayson [7] in the context of qualitative analysis of evolution of simple planar curves driven by the curvature. It has been utilized by Dziuk, Deckelnick, Mikula and Ševčovič (see e.g. [2, 3, 4, 11, 12, 13, 14, 15, 16, 17] and other references therein). The direct approach based methodology represents the flow of planar curves by evolution of its position vector x which is a solution to the geometric equation $\partial_t x = \beta \vec{N} + \alpha \vec{T}$ where $\vec{N}, \vec{T}$ are

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the unit inward normal and tangent vectors, resp. Next one can construct a closed system of parabolic-ordinary differential equations for the curvature, tangential angle, local length and position vector. Notice that other techniques used when solving the geometric equation (1.2), like e.g. level-set method due to Osher and Sethian (cf. [19, 18]) or phase-field approximations (see e.g. Beneš [1]) treat the geometric equation (1.2) by means of a solution to a higher dimensional parabolic problem. The advantage of the direct approach based method is that we have to solve one space dimensional evolutionary problems only. The second advantage consists in possibility of choosing the tangential velocity functional α significantly improving and stabilizing numerical computations as it has been shown by many authors (see e.g. [2, 8, 9, 10, 13, 14, 15, 16]). The main purpose of this paper is to study global solutions of geodesic curvature driven fbws, examine stability of closed geodesic curves. We present a criterion for stability of a closed geodesic curve and present an example of such curve on non-convex surface.

The paper is organized as follows. The next section is devoted to reduction of the geodesic curvature driven fbw of surface curves to a fbw of planar curves driven by the normal velocity v given as in (1.2). We also present a closed system of parabolic-ordinary differential equations governing the evolution of the curvature k, tangent angle ν, local length g and position vector x of the evolving family of planar curves. Next we show how to construct a suitable tangential velocity functional α yielding a robust numerical scheme for solving governing equations. Finally, we investigate global solutions and stability of stationary closed geodesic curves. Several computational examples are presented in the last section.

2. Projection of a flow of surface curves to the plane. Throughout the paper we will always assume that a surface $\mathcal{M} = \{(x, z)) \in \mathbb{R}^3,\ z = \phi(x),\ x \in \Omega\}$ is a smooth graph of a function $\phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ defined in some domain $\Omega \subset \mathbb{R}^2$. Hereafter, the symbol (x, z) stands for a vector $(x_1, x_2, z) \in \mathbb{R}^3$ where $x = (x_1, x_2) \in \mathbb{R}^2$. In such a case any smooth closed curve $\mathcal{G}$ on the surface $\mathcal{M}$ can be then represented by its vertical projection to the plane, i.e. $\mathcal{G} = \{(x, z) \in \mathbb{R}^3,\ x \in \Gamma,\ z = \phi(x)\}$ where Γ is a closed planar curve in $\mathbb{R}^2$. Recall, that for a curve $\mathcal{G} = \{(x, \phi(x)) \in \mathbb{R}^3,\ x \in \Gamma\}$ on a surface $\mathcal{M} = \{(x_1, x_2, \phi(x_1, x_2)) \in \mathbb{R}^3,\ (x_1, x_2) \in \Omega\}$ the geodesic curvature $\mathcal{K}_g$ is given by

$$\mathcal{K}_g = -\sqrt{EG - F^2}(x_1'' x_2' - x_1' x_2'' - \Gamma^2_{11} x_1'^3 + \Gamma^1_{22} x_2'^3 - (2\Gamma^2_{12} - \Gamma^1_{11}) x_1'^2 x_2' + (2\Gamma^1_{12} - \Gamma^2_{22}) x_1' x_2'^2)$$

where E, G, F are coefficients of the first fundamental form and Γ^k_{ij} are Christoffel symbols of the second kind. Here $(.)'$ denotes the derivative with respect to the unit speed parameterization of a curve on a surface. In terms of geometric quantities related to a vertically projected planar curve we obtain, after some calculations, that

$$\mathcal{K}_g = \frac{1}{\left(1 + (\nabla\phi.\vec{T})^2\right)^{\frac{3}{2}}} \left(\left(1 + |\nabla\phi|^2\right)^{\frac{1}{2}}\ k + \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{(1 + |\nabla\phi|^2)^{\frac{1}{2}}} \nabla\phi.\vec{N} \right) \tag{2.1}$$

(see [16]). Moreover, the unit inward normal vector $\vec{\mathcal{N}} \perp T_x(\mathcal{M})$ to a surface curve $\mathcal{G} \subset \mathcal{M}$ relative to $\mathcal{M}$ can be expressed as

$$\vec{\mathcal{N}} = \frac{\left((1 + (\nabla\phi.\vec{T})^2)\vec{N} - (\nabla\phi.\vec{T})(\nabla\phi.\vec{N})\vec{T},\ \nabla\phi.\vec{N}\right)}{\left((1 + |\nabla\phi|^2)(1 + (\nabla\phi.\vec{T})^2)\right)^{\frac{1}{2}}}.$$

(see also [16]). Hence for the normal velocity $\mathcal{V}$ of $\mathcal{G}_t = \{(x, \phi(x)),\ x \in \Gamma_t\}$ we have

$$\mathcal{V} = \partial_t(x, \phi(x)).\vec{\mathcal{N}} = (\vec{N}, \nabla\phi.\vec{N}).\beta\vec{\mathcal{N}} = \left(\frac{1 + |\nabla\phi|^2}{1 + (\nabla\phi.\vec{T})^2}\right)^{\frac{1}{2}} \beta$$

where β is the normal velocity of the vertically projected planar curve Γ_t having the unit inward normal $\vec{N}$ and tangent vector $\vec{T}$. Following the so-called direct approach (see c [2, 3, 4, 8, 12, 13, 14, 15, 16, 17]) the evolution of planar curves $\Gamma_t, t \geq 0,$ can be described by a solution $x = x(., t) \in \mathbb{R}^2$ to the position vector equation

$$\partial_t x = \beta\vec{N} + \alpha\vec{T} \tag{2.2}$$

where β and α are normal and tangential velocities of Γ_t, resp. Assuming the family of surface curves $\mathcal{G}_t$ satisfies (1.1) it has been shown in [16] that the geometric equation $v = \beta(x, k, \nu)$ for the normal velocity v of the vertically projected planar curve Γ_t can be written in the following form:

$$v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla\phi(x).\vec{N} \tag{2.3}$$

where $a = a(x, \nu) > 0$ and $b = b(x, \nu)$ are smooth functions given by

$$a(x, \nu) = \frac{\delta}{1 + (\nabla\phi.\vec{T})^2}\,, \quad b(x, \nu) = -a(x, \nu)\frac{\vec{T}^T\nabla^2\phi\,\vec{T}}{1 + |\nabla\phi|^2}\,, \tag{2.4}$$

where $\delta(X, \vec{\mathcal{N}}) > 0, X = (x, \phi(x)), \phi = \phi(x)$, k is the curvature, and $\vec{N} = (-\sin\nu, \cos\nu)$ and $\vec{T} = (\cos\nu, \sin\nu)$ are the curvature, unit inward normal and tangent vectors to a curve Γ_t.

In what follows, we will present a closed system of partial differential equations governing the evolution of closed planar curves driven by the normal velocity v given as in (2.3). Henceforth we will parameterize an embedded regular plane curve Γ by a smooth function $x : S^1 \to \mathbb{R}^2$, i.e. $\Gamma = \text{Image}(x) := \{x(u), u \in S^1\}$. The circle S^1 will be identified with the interval $[0, 1]$ by taking into account the periodic boundary conditions at $u = 0, 1$. The unit arc-length parameterization of a curve $\Gamma = \text{Image}(x)$ is denoted by s, $ds = g\, du$ where $g = |\partial_u x|$. Let $\vec{N}$ and $\vec{T}$ denote the unit inward normal and tangent vector to Γ, resp. By k we denote the signed curvature such that k is positive on convex sub-arcs of a closed curve. By ν we denote the tangent angle to Γ, i.e. $\nu = \arg(\vec{T})$. Then $\vec{T} = (\cos\nu, \sin\nu)$, $\vec{N} = (-\sin\nu, \cos\nu)$, and, by Frenét's formulae, $\partial_s\vec{T} = k\vec{N}$, $\partial_s\vec{N} = -k\vec{T}$ and $\partial_s\nu = k$. Let a regular smooth initial curve $\Gamma_0 = \text{Image}(x_0)$ be given. A family of planar curves $\Gamma_t = \text{Image}(x(., t)), t \in [0, T)$, satisfying (1.2) can be represented by a solution $x = x(u, t)$ to the position vector equation (2.2). Notice that $\beta = \beta(x, k, \nu)$ depends on x, k, ν and this is why we have to provide equation for the variables k, ν as well as local length $g = |\partial_u x|$, also. The governing system of equations for a general position vector equation (2.2) has been derived and analyzed by the authors in [14, 15, 16] for a wide class of normal velocities β. They are straightforward modifications of well-known geometric equations derived for the case of a zero tangential velocity α (see e.g. [6]). In the case of a nontrivial tangential velocity functional α the system of parabolic–ordinary governing equations has the following form:

$$\partial_t k = \partial_s^2\beta + \alpha\partial_s k + k^2\beta\,, \tag{2.5}$$

$$\partial_t \nu = \partial_k\beta\partial_s^2\nu + (\alpha + \partial_\nu\beta)\partial_s\nu + \nabla_x\beta.\vec{T}\,, \tag{2.6}$$

$$\partial_t g = -gk\beta + \partial_u\alpha\,, \tag{2.7}$$

$$\partial_t x = \beta\vec{N} + \alpha\vec{T} \tag{2.8}$$

where $(u,t) \in [0,1] \times (0,T)$, $ds = g\,du$, $\vec{T} = \partial_s x = (\cos\nu, \sin\nu)$, $\vec{N} = \vec{T}^\perp = (-\sin\nu, \cos\nu)$, $\beta = \beta(x,k,\nu)$. A solution (k,ν,g,x) to (2.5) – (2.8) is subject to initial conditions

$$k(.,0) = k_0\,,\ \nu(.,0) = \nu_0\,,\ g(.,0) = g_0\,,\ x(.,0) = x_0(.)$$

and periodic boundary conditions at $u = 0,1$ except of ν for which we require the boundary condition $\nu(1,t) \equiv \nu(0,t)\ \mathrm{mod}(2\pi)$. The initial conditions for k_0, ν_0, g_0 and x_0 must satisfy natural compatibility constraints: $g_0 = |\partial_u x_0| > 0\,,\ k_0 = g_0^{-3} det\left(\partial_u x_0, \partial_u^2 x_0\right)\,,\ \partial_u \nu_0 = g_0 k_0$ following from the equation $k = det\left(\partial_s x, \partial_s^2 x\right)$ and Frenét's formulae applied to the initial curve $\Gamma_0 = \mathrm{Image}(x_0)$. Notice that the system of governing equations consists of coupled parabolic-ordinary differential equations.

In [14, 15, 16, 17] we have shown the importance of a suitable choice of a tangential velocity functional α entering the governing system of equations (2.5)-(2.8). Although α occurs in the governing equations the family of planar curves $\Gamma_t = \mathrm{Image}(x(.,t)), t \in [0,T)$, is independent of a particular choice of the tangential velocity α as it does not change the shape of a curve. On the other hand, a solution k,ν,g,x to (2.5) – (2.8) depends on α. The tangential velocity α can be therefore appropriately chosen in order to stabilize a numerical scheme based on discretization of (2.5)-(2.8). It has been shown in [15, 16, 17] that if α is a solution to the equation:

$$\partial_s \alpha = k\beta - \langle k\beta\rangle_\Gamma\,, \quad \alpha(0,.) = 0\,, \tag{2.9}$$

where L is the length of the plane curve Γ and $\langle k\beta\rangle_\Gamma$ is the average of $k\beta$ over the curve Γ, i.e. $\langle k\beta\rangle_\Gamma = \frac{1}{L}\int_\Gamma k\beta\,ds$, then we obtain parameterization preserving relative local length: $g(u,t)/L_t = g(u,0)/L_0$ for all $u \in [0,1]$ and $t \in (0,T_{max})$. Construction of a suitable tangential velocity functional α leading to redistribution preserving relative local length has been discussed by Hou *et al.* in [8, 9]. It has been generalized to the case of asymptotically uniform parameterization by the authors in in [14, 15, 16]. With such a choice of a tangential velocity functional α the governing system of equations can be rewritten in the following form:

$$\partial_t k = \partial_s^2 \beta + \partial_s(\alpha k) + k\langle k\beta\rangle_\Gamma\,, \tag{2.10}$$

$$\partial_t \nu = \partial_k \beta \partial_s^2 \nu + (\alpha + \partial_\nu \beta)\partial_s \nu + \nabla_x \beta.\vec{T}\,, \tag{2.11}$$

$$\partial_t g = -g\langle k\beta\rangle_\Gamma\,, \tag{2.12}$$

$$\partial_t x = \beta\vec{N} + \alpha\vec{T}\,. \tag{2.13}$$

From the numerical approximation point of view, the principal advantage of the above governing system of equations in comparison to the system (2.5)-(2.8) is that strong reaction terms $k^2\beta$ and $k\beta$ appearing in (2.5) and (2.7) are now replaced by averaged values $k\langle k\beta\rangle_\Gamma$ and $\langle k\beta\rangle_\Gamma$ in (2.10) and (2.12), resp.

Local in-time existence of a classical solution (k,ν,g,x) to the (2.10)-(2.13) provided that the initial curve Γ_0 is smooth and regular ($g > 0$ on Γ_0 and the function $\beta = \beta(x,\nu,k)$ is sufficiently smooth has been shown in [16, Theorem 5.1]. Moreover, it has been also shown that if the maximal solution is defined on the interval $[0,T_{max})$ then we have either $T_{max} = +\infty$ or $\liminf_{t\to T_{max}^-} \min_{\Gamma_t} \partial_k\beta(x,k,\nu) = 0$ or $T_{max} < +\infty$ and $\max_{\Gamma_t}|k| \to \infty$ as $t \to T_{max}$.

3. Closed stationary curves and their stability. In what follows we analyze stationary surface curves with respect to the normal velocity $\mathcal{V} = \delta\mathcal{K}_g$, i.e. geodesic surface curves satisfying $\mathcal{K}_g = 0$. The analysis is based on results obtained by the authors in [17] for the case of the normal velocity given by $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$. In terms of the vertical projection of surface curves to the plane it suffices to investigate stability of planar stationary curves satisfying $\beta(x, k, \nu) = 0$ where β is given by (2.3). Stability results presented in [17] are given in terms of a general normal velocity function β. This why we have to verify their assumptions in the case of (2.3) representing vertical projection of the flow of surface curves driven by the geometric equation (1.2).

A closed smooth planar curve $\bar\Gamma = \text{Image}(\bar x)$ is called a stationary curve with respect to the normal velocity β iff $\beta(\bar x, \bar k, \bar\nu) = 0$ on $\bar\Gamma$ where $\bar x, \bar k$ and $\bar\nu$ are the position vector, curvature and tangential angle of the curve $\bar\Gamma$.

Since the presence of an arbitrary tangential velocity functional in the system of governing equations has no impact on the shape of evolving curves $\Gamma_t = \text{Image}(x(.,t))$ we will take $\alpha = 0$ in the analysis of stability of stationary curves. As $k = \partial_s \nu = g^{-1}\partial_u \nu$ the governing system of equations (2.5)–(2.8) can be reduced to:

$$
\begin{aligned}
&\partial_t k = g^{-1}\partial_u(g^{-1}\partial_u\beta) + k^2\beta\,, \quad && \partial_t\nu = g^{-1}\partial_u\beta\,,\\
&\partial_t g = -gk\beta\,, && \partial_t x = \beta\vec N\,,
\end{aligned}
\tag{3.1}
$$

$u \in S^1, t \in (0,T)$. Let $\bar\Gamma = \text{Image}(\bar x)$ be a stationary curve having the curvature $\bar k$, tangential angle $\bar\nu$, the local length $\bar g$, position vector $\bar x$ and the unit normal vector $\bar{\vec N}$. In order to analyze stability of $\bar\Gamma$ we have to investigate the behavior of infinitesimal variations of k, ν, g and x. Variations from a steady state $(\bar k, \bar\nu, \bar g, \bar x)$ will be denoted by $(\gamma^k, \gamma^\nu, \gamma^g, \gamma^x)$. Since $\bar\beta = \beta(\bar x, \bar k, \bar\nu) = 0$ on $\bar\Gamma$ we have $\partial_u\bar\beta = \partial_u^2\bar\beta = 0$ on $\bar\Gamma$. Hence infinitesimally small variations $\gamma^k, \gamma^\nu, \gamma^g$ and γ^x satisfy the linearized system

$$
\begin{aligned}
&\partial_t\gamma^k = \bar g^{-1}\partial_u(\bar g^{-1}\partial_u\gamma^\beta) + \bar k^2\gamma^\beta\,, \quad && \partial_t\gamma^\nu = \bar g^{-1}\partial_u\gamma^\beta\,,\\
&\partial_t\gamma^g = -\bar g\bar k\gamma^\beta\,, && \partial_t\gamma^x = \gamma^\beta\bar{\vec N}
\end{aligned}
\tag{3.2}
$$

for $u \in S^1, t > 0$. Here $\gamma^\beta = \beta(\bar x + \gamma^x, \bar k + \gamma^k, \bar\nu + \gamma^\nu) - \beta(\bar x, \bar k, \bar\nu) = \nabla_x\bar\beta.\gamma^x + \partial_k\bar\beta\gamma^k + \partial_\nu\bar\beta\gamma^\nu + h.o.t.$ Clearly, all variations $\gamma^k, \gamma^\nu, \gamma^g, \gamma^x, \gamma^\beta$ are subject to periodic boundary conditions at $u = 0, 1$. As $\nabla_x\bar\beta = \nabla_x\beta(\bar x, \bar k, \bar\nu)$, $\partial_k\bar\beta = \partial_k\beta(\bar x, \bar k, \bar\nu)$, and $\partial_\nu\bar\beta = \partial_\nu\beta(\bar x, \bar k, \bar\nu)$ do not depend on time the total variation γ^β satisfies the scalar parabolic equation

$$
\begin{aligned}
\partial_t\gamma^\beta &= \nabla_x\bar\beta.\partial_t\gamma^x + \partial_k\bar\beta\partial_t\gamma^k + \partial_\nu\bar\beta\partial_t\gamma^\nu\\
&= \partial_k\bar\beta\bar g^{-1}\partial_u(\bar g^{-1}\partial_u\gamma^\beta) + \partial_\nu\bar\beta\bar g^{-1}\partial_u\gamma^\beta + (\partial_k\bar\beta\bar k^2 + \nabla_x\bar\beta\bar{\vec N})\gamma^\beta\,,
\end{aligned}
\tag{3.3}
$$

i.e. $\partial_t\gamma^\beta = P\partial_u^2\gamma^\beta + R\partial_u\gamma^\beta + Q\gamma^\beta$ where

$$
P = \bar g^{-2}\partial_k\bar\beta\,, \quad R = \bar g^{-1}\partial_\nu\bar\beta + \bar g^{-1}\partial_k\bar\beta\partial_u\bar g^{-1}\,, \quad Q = \partial_k\bar\beta\bar k^2 + \nabla_x\bar\beta.\bar{\vec N}\,. \tag{3.4}
$$

Functions P, Q and R are 1-periodic in u variable and depend on the stationary curve $\bar\Gamma$ only. A solution γ^β to (3.3) is subject to periodic boundary conditions at $u = 0, 1$.

Our concept of stability of stationary curves is based on the stability of a trivial solution $\gamma = 0$ to the initial-periodic boundary value problem for equation (3.3). Stability of a trivial solution to (3.3) can be described by the largest eigenvalue λ_1 of the linear operator A defined by the right hand side of (3.3), i.e.

$$
A\psi := P\psi'' + R\psi' + Q\psi \tag{3.5}
$$

subject to periodic boundary conditions $\psi(0) = \psi(1), \psi'(0) = \psi'(1)$. We will say that a stationary closed curve $\bar\Gamma$ is *linearly stable* if $\lambda_1 < 0$. It is called *unstable* if $\lambda_1 > 0$. According to [17, Lemma 3.1] the linear operator $A : D(A) \subset L^2(S^1, w) \to L^2(S^1, w), D(A) =$

$W^{2,2}(S^1)$, is a self-adjoint operator in the weighted Lebesgue space $L^2(S^1, w)$ provided that the weight $w(u) = P(u)^{-1} \exp(\int_0^u \frac{R(v)}{P(v)} dv)$ is 1-periodic in the u variable. Latter condition is satisfied if and only if

$$\int_{\bar\Gamma} \frac{\partial_\nu \bar\beta}{\partial_k \bar\beta}\, ds = 0\,,. \tag{3.6}$$

According to [17, Definition 3.3] a normal velocity function $\beta = \beta(x, k, \nu)$ satisfying such a condition on any closed stationary curve $\bar\Gamma$ is called an admissible normal velocity. In what follows, we will prove that the normal velocity given by (2.3) indeed satisfies the admissibility condition (3.6). In order to verify (3.6), similarly as in the proof of [17, Prop. 3.2], we denote

$$d := \frac{\vec T^T \nabla^2\phi\, \vec T}{1+|\nabla\phi|^2} \nabla\phi.\vec N \quad \text{and} \quad h := \partial_s \phi = \nabla\phi.\vec T\,. \tag{3.7}$$

Let $\bar\Gamma$ be a stationary curve with respect to β. Then $\beta(x, k, \nu) = a(k + d)$ and thus $k + d = 0$ on $\bar\Gamma$. Moreover,

$$\frac{\partial_\nu \beta}{\partial_k \beta} = \frac{\partial_\nu a}{a}(k + d) + \partial_\nu d = \partial_\nu d \tag{3.8}$$

on $\bar\Gamma$. It follows from (3.7) and the identities $k + d = 0$ and $1 + |\nabla\phi|^2 = 1 + (\nabla\phi.\vec T)^2 + (\nabla\phi.\vec N)^2 = 1 + h^2 + (\nabla\phi.\vec N)^2$ that

$$\begin{aligned} (1+h^2)^{\frac32}\partial_s\left(\frac{h}{(1+h^2)^{\frac12}}\right) &= \vec T^T\nabla^2\phi\,\vec T + k\nabla\phi.\vec N \\ = \vec T^T\nabla^2\phi\,\vec T\left(1 - \frac{(\nabla\phi.\vec N)^2}{1+|\nabla\phi|^2}\right) &= \frac{1+h^2}{1+|\nabla\phi|^2} T^T\nabla^2\phi\,\vec T \end{aligned}$$

on $\bar\Gamma$. Since $\partial_\nu d = \left(2T^T\nabla^2\phi\,\vec N(\nabla\phi.\vec N) - T^T\nabla^2\phi\,\vec T(\nabla\phi.\vec T)\right)/(1+|\nabla\phi|^2)$ and

$$\partial_s \ln(1+|\nabla\phi|^2) = \frac{2\nabla\phi\nabla^2\phi\,\vec T}{1+|\nabla\phi|^2} = \frac{2}{1+|\nabla\phi|^2}\left(T^T\nabla^2\phi\,\vec T(\nabla\phi.\vec T) + T^T\nabla^2\phi\,\vec N(\nabla\phi.\vec N)\right)$$

we obtain from (3.8) and (3.9) that the following identity is satisfied on any stationary curve $\bar\Gamma$:

$$\begin{aligned} \frac{\partial_\nu\beta}{\partial_k\beta} &= \partial_s \ln(1+|\nabla\phi|^2) - 3h\frac{T^T\nabla^2\phi\,\vec T}{1+|\nabla\phi|^2} \\ &= \partial_s \ln(1+|\nabla\phi|^2) - 3h(1+h^2)^{\frac12}\partial_s\left(\frac{h}{(1+h^2)^{\frac12}}\right) \\ &= \partial_s \ln(1+|\nabla\phi|^2) - 3\partial_s \ln\sqrt{1+h^2} = \partial_s\left(\ln\frac{1+|\nabla\phi|^2}{(1+h^2)^{\frac32}}\right) \end{aligned}$$

Hence $\int_{\bar\Gamma} \frac{\partial_\nu\beta}{\partial_k\beta} ds = 0$, as claimed.

It is worthwhile noting that the first eigenvalue λ_1 is negative and consequently the stationary curve $\bar\Gamma$ is linearly stable in the case $\sup_{\bar\Gamma} Q < 0$ (cf. [17, Corollary 3.2]).

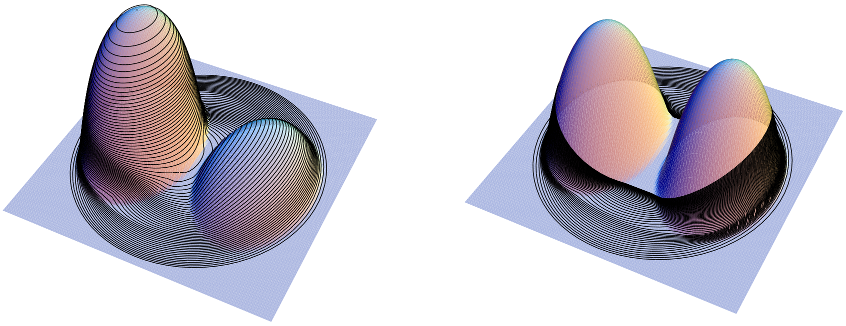

FIG. 4.1. *A geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps having different heights (left). A flow approaching a linearly stable closed geodesic curve on a surface with two sufficiently high humps (right)*

4. Examples. Our numerical scheme for solving the governing system of equations (2.10) - (2.10) has been described in [16] for a general class of normal velocities of the form $\beta(x, \nu, k) = a(x, \nu)k + c(x, \nu)$. The reader is referred to [16, Section 6] for further details of numerical approximation. We only recall that the scheme is semi-implicit in time. It means that all non-linear terms in equations are treated from the previous time step whereas linear terms are solved at the current time level. Such a discretization leads to a solution of linear systems of equations at every discrete time level. At the j-th time step, $j = 1, .., m$, we first find discrete values of the tangential velocity functional α. Then the values of a redistribution parameter are computed and subsequently utilized for updating discrete local lengths g. Using already computed local lengths, tridiagonal systems with periodic boundary conditions are constructed and solved for discrete curvature k, tangent angle ν and, finally for discrete values of the position vector x. Detailed derivation and discussion of the numerical scheme based on the so-called fbwing finite volume method is given in [16]. It was also shown that the experimental order of convergence of this scheme is at least one which is a natural order in the case of finite volume approximations.

In the rest of this section we present several examples illustrating a geodesic fbw $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps. In Fig. 4.1 (left) we considered a surface $\mathcal{M}$ defined as a graph of the function $\phi(x) = f(x_1 - 1, x_2) + 3f(x_1 + 1, x_2)$ where $f(x) = 2^{-1/(1-|x|^2)}$ for $|x| < 1$ and $f(x) = 0$ for $|x| \geq 1$ is a smooth bump function. In this example the evolving family of surface curves shrinks to a point in finite time. On the other hand, in Fig. 4.1 (right) we considered the function $\phi(x) = 3(f(x_1 - 1, x_2) + f(x_1 + 1, x_2))$. We took the time step $\tau = 0.0002$. As an initial curve we chose an ellipse centered at the origin with axes 2 and $\sqrt{2}$. The spatial mesh contained 400 grid points. The initial curve was evolved until the time $T = 13$. As it can be seen from Fig. 4.1 the evolving family of surface curves approaches a closed geodesic curve $\bar{\Gamma}$ as $t \to \infty$. It is worth to note that $\sup_{\bar{\Gamma}} Q = 0.000275 > 0$ and therefore the simple stability criterion $\sup_{\bar{\Gamma}} Q < 0$ can not be used and we had to compute the first (largest) eigenvalue of the Sturm-Liouville operator A (see (3.5)). It turns out that for the largest eigenvalue λ_1 we have $\lambda_1 \approx -0.095$ and therefore the stationary curve $\bar{\Gamma}$ is linearly stable. In Fig. 4.2 we extend the fbw form Fig. 4.1 to the case of a closed non-convex surface. Such an extension is treatable by our theory because the fbw of surface curves takes place only in one map of the surface atlas. This example indicates existence of stable geodesic curves on two dimensional non-convex surfaces with simple genus.

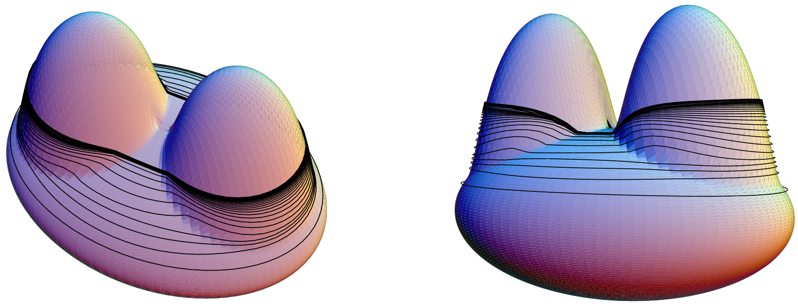

FIG. 4.2. *Two different view points of a geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a closed non-convex surface with two equally high humps. A flow approaching a linearly stable closed geodesic curve.*

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