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SUPPORT SPLITTING, CONNECTING, AND RE-SPLITTING PHENOMENA IN THE FLOW THROUGH AN ABSORBING MEDIUM

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Abstract. Mathematical models for an interaction between diffusion and absorption exhibit a wide variety of wave phenomena in the several fields. A representative model is given in the form of the description of the flow through an absorbing porous medium. The most striking property caused by the interaction is the occurrence of *support re-splitting phenomena*, where the support means the region occupied by the flow. In this paper the mathematical justification of such phenomena is stated.

Key words. Nonlinear diffusion, porous media equation, finite extinction, interfaces, support splitting, difference scheme

AMS subject classifications. 65M12, 35K65, 35B99

1. Introduction. We consider the flow of the liquids through a one-dimensional homogeneous porous medium with absorption, which is represented in the form of the initial value problem:

$$v_t = (v^m)_{xx} - cv^p, \qquad x \in \mathbf{R}^1, \quad t > 0, \tag{1.1}$$

$$v(0,x) = v^0(x), \qquad x \in \mathbf{R}^1, \tag{1.2}$$

where v and $-cv^p$ denote the density in the flow of the liquids and the volumetric absorption, respectively. Here we have the following assumptions:
(i) $m(> 1)$, $p(> 0)$, and $c(\geq 0)$ are constants and $m + p \geq 2$;
(ii) $v^0(x) \in C^0(\mathbf{R}^1)$ is nonnegative and has compact support.
From analytical points of view, Aronson [1], Oleinik, Kalashnikov and Chzou Yui-Lin [12], Kalashnikov [7, 8], and Herrero and Vázquez [6] proved the existence and uniqueness of a weak solution and the property of the finite propagation of the support which is caused by the degeneracy of the diffusion rate at points where $v = 0$. Moreover, $v(t,x)$ is smooth in the open set $\mathcal{P}(v) = \{(t,x) | v(t,x) > 0 \text{ and } t > 0\}$, and has the following properties:
(P-1) For $c = 0$, or $c > 0$ and $p \geq 1$ the diffusion is active and supp $v(t,\cdot)$ monotonously expands as t increases;
(P-2) For $c > 0$ and $0 < p < 1$ the absorption is active and the solution vanishes identically at some finite time $T^* > 0$.

In Case (P-1) supp $v(t,\cdot)$ never splits into any multiple connected components for $t > 0$, when supp $v^0(x)$ is connected. Thus the *support splitting phenomena* never appear. In Case (P-2) there is a possibility of the support to split, when $v^0(x)$ has two local maxima (see Fig. 1.1). Rosenau and Kamin [13] suggested this possibility by numerical computation. Chen, Matano and Mimura [3] constructed the initial

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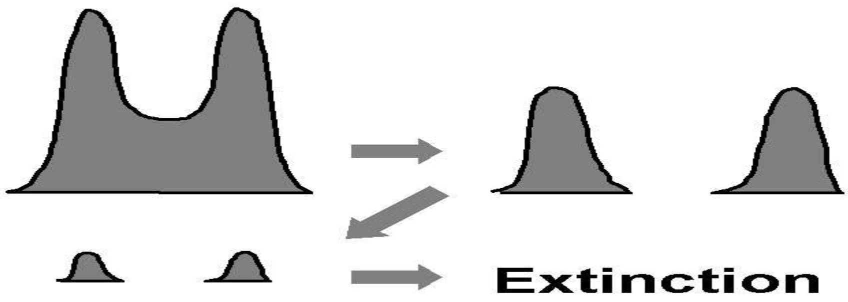

FIG. 1.1. *Support splitting phenomena.*

value for which the support of the solution splits into multiple connected components in a finite time. This motivates us to investigate the detail of the behavior of the support. We continued numerical computation and found the following *support re-splitting phenomena*.

(S-1) After *support splitting phenomena* appear, the support becomes connected, and thereafter *support splitting phenomena* appear again (see Fig. 1.2 and 1.3).

We note that the phenomenon (S-1) includes the following process.

(S-2) The support consisting of two connected components, while it is initially disconnected, becomes connected, and thereafter *support splitting phenomena* appear.

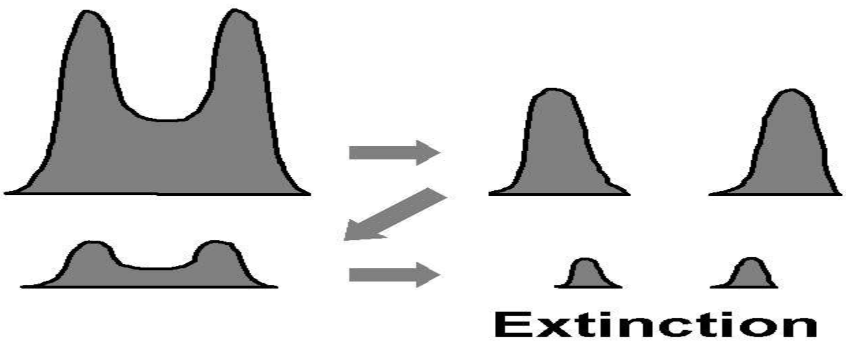

FIG. 1.2. *Support re-splitting phenomena.*

In this paper, we show the construction of the initial value for which the phenomenon (S-1) appears under the following assumption.

ASSUMPTION A. $c > 0$, $m + p = 2$ and $0 < p < 1$.

Our proof is based on the finite difference scheme([9, 10, 11]) and the comparison theorem ([2]). Unfortunately, in the case where $m + p \neq 2$, $m > 1$ and $0 < p < 1$, we are unable to succeed in constructing the finite difference scheme with convergence. This is the reason why we are concerned with the specific case stated in Assumption A.

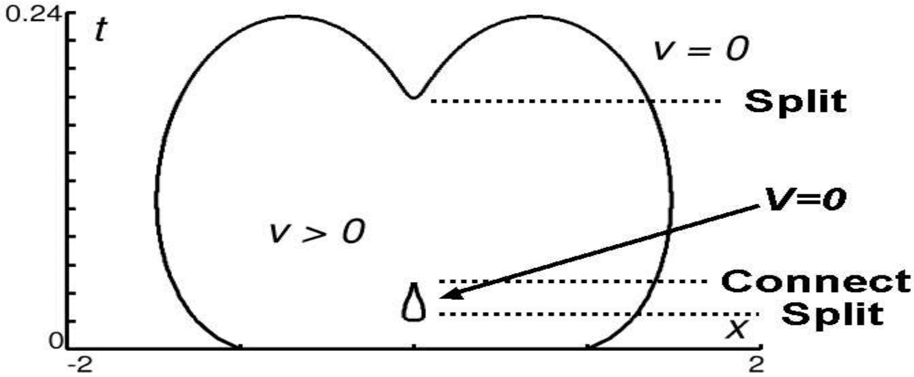

Fig. 1.3. *Numerical support in re-splitting phenomena, where* $m = 1.5$, $p = 0.5$ *and* $c = 5$.

2. Finite difference schemes. We put $u = v^{m-1}$ and rewrite (1.1)–(1.2) as follows:

$$u_t = muu_{xx} + a(u_x)^2 - c', \tag{2.1}$$

$$u(0,x) = u^0(x) \equiv (v^0(x))^{m-1}, \tag{2.2}$$

where $a = \dfrac{m}{m-1}$, $c' = (m-1)c$ and the term of absorption is written as the constant $-c'$ by the assumption $m + p = 2$. Our scheme approximates the problem (2.1)–(2.2) instead of (1.1)–(1.2). Let h be a space mesh width and V_h be the set of the nonnegative and piecewise-linearly interpolated functions $u_h = u_h(x)$ with the mesh $\mathcal{M}_h = \{\ell, Lh, (L+1)h, \cdots, (R-1)h, Rh, r\}$, where the L and R are integers, and ℓ and r denote the left and right interfaces of u_h, respectively. The scheme is described as follows:

Find the sequence $\{u_h^n\}_{n=1,2,\cdots} \in V_h$ with the mesh $\mathcal{M}_h^n = \{\ell_n, L_nh, (L_n + 1)h, \cdots, (R_n - 1)h, R_nh, r_n\}$ for each $u_h^0 \in V_h$ such that

$$u_h^{n+1} = S_{h,k}u_h^n \quad \text{for } n = 0, 1, 2, \cdots, \tag{2.3}$$

where $u_h^0(x) = u^0(x)$ on $\mathcal{M}_h^0$. $S_{h,k}$ is somewhat complicated form and its detail is stated in [9, 10, 11]. We omit the description of $S_{h,k}$. The variable time step $k = k_{n+1} \equiv t_{n+1} - t_n$ ($t_0 = 0$) is determined by

$$k = \frac{1}{c'}\max(u_L, u_{L+1}) \text{ for the approximation to the left interface, or} \tag{2.4}$$

$$k = \frac{1}{c'}\max(u_R, u_{R-1}) \text{ for the approximation to the right interface.} \tag{2.5}$$

When $S_{h,k}u_h^{n^*} \equiv 0$ holds for some integer $n^* > 0$, we put the numerical extinction time $T_h^* = t_{n^*+1} \equiv t_{n^*} + k_{n^*+1}$, and stop the numerical computation. We define the left(resp. right) numerical interface curves $\ell_h(t)$(resp. $r_h(t)$) by piecewise-linearly interpolating (t_n, ℓ_n)(resp.(t_n, r_n))$(0 \le n \le n^*)$. We state several results without proof, which play an important role in constructing the initial value for which the phenomenon (S-1) appears. For this end we introduce the following

Condition B. i) $v^0(x) \in C^0(\mathbf{R}^1)$ is a nonnegative function with compact support and $\left((v^0(x))^{m-1}\right)_x \in L^\infty(\mathbf{R}^1) \cap BV(\mathbf{R}^1)$;

ii) $\left((v^0(x))^{m-1}\right)_x$ is absolutely continuous on $\mathbf{I} = \{x | v^0(x) > 0\}$ and $\text{ess.inf}_{\mathbf{I}} \left((v^0(x))^{m-1}\right)_{xx}$ is finite.

We define the constants $C_j(v^0)$ $(j = 0, 1, 2)$ by

$$\left\{ \begin{array}{l} C_0(v^0) = \|(v^0)^{m-1}\|_\infty, \quad C_1(v^0) = \left\|\left((v^0)^{m-1}\right)_x\right\|_\infty, \\ C_2(v^0) = -\text{ess.inf}_{\mathbf{I}} \left((v^0(x))^{m-1}\right)_{xx}, \end{array} \right. \tag{2.6}$$

where $\|\cdot\|_\infty$ denotes $\|\cdot\|_{L^\infty(\mathbf{R}^1)}$.

Theorem 2.1 (Basic estimates [9],[11]). *Let $u_h^0 \in V_h$. Then u_h^n either becomes extinct or belongs to V_h for each $n \geq 0$, and the following estimates hold for all $n \geq 0$:*

$$T_h^* \leq t_n + \frac{\|u_h^n\|_\infty}{c'}, \tag{2.7}$$

$$0 \leq r_n - \ell_n \leq (r_0 - \ell_0 + 2a\|(u_h^0)_x\|_\infty t_n) \ \textit{if} \ u_h^n \not\equiv 0, \tag{2.8}$$

$$0 \leq u_h^n(x) \leq \max(\|u_h^0\|_\infty - c't_n, 0) \quad \textit{on} \ \mathbf{R}^1, \tag{2.9}$$

$$\|(u_h^n)_x\|_\infty \leq \|(u_h^0)_x\|_\infty, \tag{2.10}$$

$$TV((u_h^n)_x) \leq TV((u_h^0)_x), \tag{2.11}$$

$$\begin{aligned} \|(u_h^{n+1} - u_h^n)/k_{n+1}\|_{L^1(\mathbf{R}^1)} &\leq (m+a)\|u_h^0\|_\infty TV((u_h^0)_x) \\ &\quad + c'(r_0 - \ell_0 + 2a\|(u_h^0)_x\|_\infty t_n), \end{aligned} \tag{2.12}$$

$$\inf_{i \in \mathbf{Z}} \delta^2 u_i^0 \leq \inf_{i \in \mathbf{Z}} \delta^2 u_i^n, \tag{2.13}$$

where $\delta^2 u$ denotes a usual finite difference approximation to u_{xx}.

Theorem 2.2 (Convergence of numerical solutions [11]). *Under Condition B let $\{h\}$ be an arbitrary sequence which tends to zero. Then, there exists the unique weak solution v of (1.1)–(1.2), and*

$$\|v_h - v\|_{L^\infty(\mathcal{H})} \longrightarrow 0 \quad \textit{and} \quad |T_h^* - T^*| \longrightarrow 0 \quad \textit{as} \quad h \to 0, \tag{2.14}$$

where $\mathcal{H} = [0, \infty) \times \mathbf{R}^1$, $v_h = (u_h)^{1/(m-1)}$, $u_h(t,x) = u_h^n(x)$ on $[t_n, t_{n+1}) \times \mathbf{R}^1$ for all t_n and h, and T^* is the extinction time.

Then, from Theorems 2.1 and 2.2 and the fact that $v(t,x)$ is smooth on $\mathcal{P}(v)$ we have

Lemma 2.3 (Basic estimates). *Assume Condition B. Then*

$$0 \leq u(t,\cdot) \leq \max(\|u^0\|_\infty - c't, 0) \quad \textit{on} \ \mathbf{R}^1, \tag{2.15}$$

$$\|u_x(t,\cdot)\|_\infty \leq \|u_x^0\|_\infty, \tag{2.16}$$

$$\int_{b_1}^{b_2} |u_{xx}(t,x)| dx = TV(u_x(t,\cdot)) \leq TV((u^0)_x) \quad \textit{for all } t \textit{ and intervals } [b_1, b_2] \subset \mathcal{P}(u), \tag{2.17}$$

$$\text{ess.inf}_{\mathbf{I}} \ u_{xx}^0 \leq \ u_{xx}(t,x) \quad \textit{for} \ \ (t,x) \in \mathcal{P}(u). \tag{2.18}$$

Theorem 2.4 (Convergence of numerical interface curves [10]). *Under Condition B let there exist a positive constant M such that*

$$((v^0)^{m-1})_x(\ell_0+0),\ -((v^0)^{m-1})_x(r_0-0)>M. \tag{2.19}$$

Let $M'(<M)$ be an arbitrary positive number. Then, $\ell_h(t)$(resp. $r_h(t)$) converges uniformly to the exact left(resp. right) interface curve $\ell(t)$(resp. $r(t)$) on $[0,T]$ as h tends to zero for each fixed $T<T(M',v^0)$, where

$$T(M',v^0)=\frac{(M-M')M'}{(2a+m)C_1(v^0)C_2(v^0)M'+3c'C_2(v^0)}. \tag{2.20}$$

Moreover,

$$(v^{m-1})_x(t,\ell(t)+0),\ -(v^{m-1})_x(t,r(t)-0)>M' \text{ on } [0,\ T(M',v^0)), \tag{2.21}$$

and

$$\dot{\ell}(t)=-a(v^{m-1})_x(t,\ \ell(t)+0)+\frac{c'}{(v^{m-1})_x(t,\ \ell(t)+0)} \quad \text{and} \tag{2.22}$$

$$\dot{r}(t)=-a(v^{m-1})_x(t,\ r(t)-0)+\frac{c'}{(v^{m-1})_x(t,\ r(t)-0)} \tag{2.23}$$

hold a.e. in $[0,T(M',v^0))$.

Theorem 2.5 (Support splitting phenomena [11]). *Assume Condition B. For $\alpha_1<\beta_1<\gamma_1<\gamma_2<\beta_2<\alpha_2$ let $v^0(x)$ satisfy*

$$v^0(x)>0 \text{ on } (\alpha_1,\ \alpha_2), \quad [\alpha_1,\ \alpha_2]=\operatorname{supp} v^0(x), \quad \text{and} \tag{2.24}$$

$$\frac{(v^0(\beta_j))^{m-1}}{c'+mC_0C_2}>\frac{\|(v^0)^{m-1}\|_{L^1[\gamma_1,\gamma_2]}}{c'(\gamma_2-\gamma_1)-(m+a)C_0TV\left(((v^0)^{m-1})_x\right)}>0 \quad (j=1,2), \tag{2.25}$$

where $C_j=C_j(v^0)$ $(j=0,2)$ are given by (2.6). Then there exist $\tilde{t}>0$ and $\tilde{x}\in[\gamma_1,\ \gamma_2]$ such that $v(\tilde{t},\tilde{x})=0$ and $v(\tilde{t},\beta_j)>0$ $(j=1,2)$ hold.

Remark 2.1. *Instead of (2.25) we assume*

$$\frac{(v^0(\beta_j))^{m-1}}{c'+mC_0C_2}>\frac{\varepsilon^{m-1}}{c'} \ (j=1,2) \ \text{ and } \ v^0(x)=\varepsilon \text{ on } [\gamma_1,\gamma_2], \tag{2.26}$$

where ε is some positive constant. Since C_j $(j=0,1,2)$ and $TV\Big(\big((v^0)^{m-1}\big)_x\Big)$ are independent of $d\equiv\gamma_2-\gamma_1$, we can take d sufficiently large so that

$$\frac{(v^0(\beta_j))^{m-1}}{c'+mC_0C_2}>\frac{\varepsilon^{m-1}}{c'-\frac{(m+a)C_0TV\left(((v^0)^{m-1})_x\right)}{\gamma_2-\gamma_1}}>\frac{\varepsilon^{m-1}}{c'} \quad (j=1,2), \tag{2.27}$$

which implies (2.25).

3. Support re-splitting phenomena.

3.1. Main result. First, we introduce two nonnegative functions $\phi(x;\varepsilon)$ and $\psi(x;\varepsilon,d_1,d_2)$ for arbitrary positive numbers ε, d_1 and d_2, which satisfy the following

Condition C. i) $\phi(x;\varepsilon)$ satisfies Conditions B with $v^0(x)=\phi(x;\varepsilon)$ and supp $\phi=[0,\ \alpha\ ]$;

ii) $\phi(x;\varepsilon)$ takes the unique local maximum at $x=\beta$, and

$$\phi(x;\varepsilon)=\varepsilon \quad \text{on} \quad [\xi,\ \gamma\ ], \tag{3.1}$$

where $0<\xi<\gamma<\beta<\alpha$;

iii)

$$\psi(x;\varepsilon,d_1,d_2)=\begin{cases} 0 & \text{if} \quad -\infty<x<d_1, \\ \phi(x-d_1;\varepsilon) & \text{if} \quad d_1<x<\xi+d_1, \\ \varepsilon & \text{if} \quad \xi+d_1<x<\gamma+d_1+d_2, \\ \phi(x-d_1-d_2;\varepsilon) & \text{if} \quad \gamma+d_1+d_2<x. \end{cases} \tag{3.2}$$

Putting

$$v^0(x;\varepsilon,d_1,d_2)=\psi(x;\varepsilon,d_1,d_2)+\psi(-x;\varepsilon,d_1,d_2) \quad \text{on } \mathbf{R}^1, \tag{3.3}$$

we choose ε, d_1 and d_2 so that the phenomenon (S-2) appears for the initial value $v^0(x;\varepsilon,d_1,d_2)$.

Next, for $v^0(x;\varepsilon,d_1,d_2)$ we introduce the initial value $v^0_\rho(x;\varepsilon,d_1,d_2)$ $(0<\rho<\varepsilon)$ satisfying

Condition D. i) $v^0_\rho(x;\varepsilon,d_1,d_2)=v^0_\rho(-x;\varepsilon,d_1,d_2)$ and $v^0(x;\varepsilon,d_1,d_2)\le v^0_\rho(x;\varepsilon,d_1,d_2)$ hold on $\mathbf{R}^1$;

ii)

$$v^0_\rho(x;\varepsilon,d_1,d_2)=\begin{cases} v^0(x;\varepsilon,d_1,d_2) & \text{if} \quad x\le -d_1-\eta, \text{ or } d_1+\eta\le x \\ \rho & \text{if} \quad -d_1\le x\le d_1, \end{cases} \tag{3.4}$$

where $0<\eta<\xi$, and $v^0_\rho(x;\varepsilon,d_1,d_2)$ decreases on $[-d_1-\eta,d_1]$ and increases on $[d_1,d_1+\eta]$;

iii) $v^0_{\rho'}(x;\varepsilon,d_1,d_2)\le v^0_\rho(x;\varepsilon,d_1,d_2)$ holds for $\rho'\le\rho$;

iv) v^0_ρ satisfies Condition B with $v^0(x)=v^0_\rho(x;\varepsilon,d_1,d_2)$ and

$$\begin{cases} \|u^0_{\rho x}(\cdot;\varepsilon,d_1,d_2)\|_\infty\le\|u^0_x(\cdot;\varepsilon,d_1,d_2)\|_\infty, \\ TV(u^0_{\rho x}(\cdot;\varepsilon,d_1,d_2))\le\big(TV(u^0_x(\cdot;\varepsilon,d_1,d_2)\big), \\ \text{ess.inf } u^0_{\rho xx}(\cdot;\varepsilon,d_1,d_2)\ge\text{ess.inf } u^0_{xx}(\cdot;\varepsilon,d_1,d_2), \end{cases} \tag{3.5}$$

where $u^0(x;\varepsilon,d_1,d_2)=(v^0(x;\varepsilon,d_1,d_2))^{m-1}$ and $u^0_\rho(x;\varepsilon,d_1,d_2)=(v^0_\rho(x;\varepsilon,d_1,d_2))^{m-1}$ (see Fig. 3.1).

Taking the constant ρ sufficiently small, we can show that the phenomenon (S-1) appears in the behavior of supp $v_\rho(t,x;\varepsilon,d_1,d_2)$, where $v_\rho(t,x;\varepsilon,d_1,d_2)$ is the solution of (1.1) with $v(0,x)=v^0_\rho(x;\varepsilon,d_1,d_2)$. This is our strategy. We state our results.

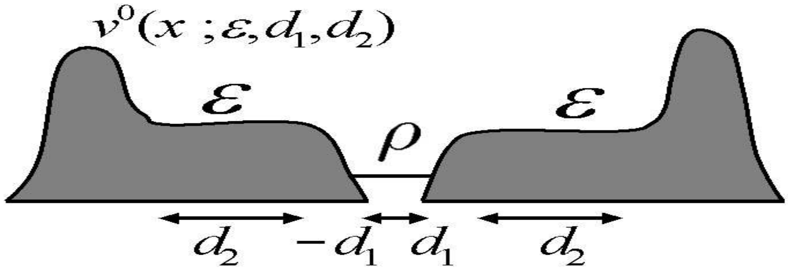

FIG. 3.1. *Initial value* $v_\rho^0(x;\varepsilon,d_1,d_2)$.

Theorem 3.1. *Let Condition C be satisfied. Suppose there exist positive constants* M *and* M' $(M>M')$ *such that*

$$\lim_{x\to+0}\left((\phi(x;\varepsilon))^{m-1}\right)_x > M > M' > 0, \quad M' > \sqrt{\frac{c'}{a}}, \tag{3.6}$$

$$\frac{(\phi(\beta;\varepsilon))^{m-1}}{c'+mC_0(\phi)C_2(\phi)} > \frac{\varepsilon^{m-1}}{c'}, \tag{3.7}$$

where $C_j(\phi)$ $(j=0,2)$ *are given by (2.6) with* $v^0=\phi$. *Then, for sufficiently small* d_1 *and sufficiently large* d_2 *there exist constants* T_1, $T_2(T_2>T_1>0)$ *and* $\tilde{x}$ *such that* supp $v(T_1,\cdot\,;\varepsilon,d_1,d_2)$ *is connected and*

$$v(T_2,\tilde{x};\varepsilon,d_1,d_2)=0 \text{ and } v(T_2,(-1)^j\beta+(-1)^j(d_1+d_2);\varepsilon,d_1,d_2)>0 \ (j=1,2), \tag{3.8}$$

where $v(t,x;\varepsilon,d_1,d_2)$ *is the solution of (1.1) with* $v(0,x)=v^0(x;\varepsilon,d_1,d_2)$ *given by (3.3).*

This theorem implies the appearance of the phenomenon (S-2).

Main Theorem (Support re-splitting phenomena)**.** *Let the initial value* $v^0(x;\varepsilon,d_1,d_2)$ *and the constant* T_1 *satisfy the conclusion of Theorem 3.1. Assume that* $v_\rho^0(x;\varepsilon,d_1,d_2)$ *satisfies Condition D. Then for sufficiently small* $\rho>0$, *there exist constants* T_0, $T_2($ $T_2>T_1>T_0>0)$, $\hat{x}$ *and* $\tilde{x}$ *such that* supp $v_\rho(T_1,\cdot\,;\varepsilon,d_1,d_2)$ *is connected and*

$$\begin{aligned}&v_\rho(T_0,\hat{x};\varepsilon,d_1,d_2)=0 \text{ and}\\&v_\rho(T_0,(-1)^j\beta+(-1)^j(d_1+d_2);\varepsilon,d_1,d_2)>0 \ (j=1,2),\end{aligned} \tag{3.9}$$

$$\begin{aligned}&v_\rho(T_2,\tilde{x};\varepsilon,d_1,d_2)=0 \text{ and}\\&v_\rho(T_2,(-1)^j\beta+(-1)^j(d_1+d_2);\varepsilon,d_1,d_2)>0 \ (j=1,2),\end{aligned} \tag{3.10}$$

where $v_\rho(t,x;\varepsilon,d_1,d_2)$ *is the solution of (1.1) with* $v(0,x)=v_\rho^0(x;\varepsilon,d_1,d_2)$.

Thus the appearance of the phenomenon (S-1) follows from this theorem.

3.2. Proof of Theorem 3.1. We first show that supp $v(t,\cdot\,;\varepsilon,d_1,d_2)$ becomes connected in a finite time. For this end let $v_{R,d_1,d_2}(t,x)$ be the solution of (1.1) with initial value $v_{R,d_1,d_2}(0,x)=v^0_{R,d_1,d_2}(x)\equiv\psi(x;\varepsilon,d_1,d_2)$. For simplicity we put

$$\begin{aligned}&u(t,x;\varepsilon,d_1,d_2)=(v(t,x;\varepsilon,d_1,d_2))^{m-1},\ u^0(x;\varepsilon,d_1,d_2)=(v^0(x;\varepsilon,d_1,d_2))^{m-1}\\&u_{R,d_1,d_2}(t,x)=(v_{R,d_1,d_2}(t,x))^{m-1}.\end{aligned}$$

We note that the number of local maximum points of $v_{R,d_1,d_2}(t,x)$ is nonincreasing. This result is stated in Proposition 2.4 by Chen, Matano and Mimura [3]. We apply this idea to the proof. Since the initial value $v^0_{R,d_1,d_2}(x)$ has *one* local maximum point at $x = d_1+d_2+\beta$ and its support consists of *one* connected component $[d_1, \alpha+d_1+d_2]$, the number of connected components of supp $v_{R,d_1,d_2}(t,x)$ never exceeds *one*.

Let $\ell_{R,d_1,d_2}(t)$ be the left interface curve of $v_{R,d_1,d_2}(t,x)$, which emanates from the point $x = d_1$. By Theorem 2.4 and (3.6) we have

$$\begin{aligned}\ell_{R,d_1,d_2}(t) &= \ell_{R,d_1,d_2}(0)\\ &\quad + \int_0^t \left\{ -a(u_{R,d_1,d_2})_x(t,\ell_{R,d_1,d_2}(t)+0) + \frac{c'}{(u_{R,d_1,d_2})_x(t,\ell_{R,d_1,d_2}(t)+0)} \right\} dt\\ &< d_1 - \left(aM' - \frac{c'}{M'} \right) t \quad \text{for} \quad t < T(M', v^0_{R,d_1,d_2}),\end{aligned} \tag{3.11}$$

where $T(M', v^0_{R,d_1,d_2})$ is given by (2.20) with $v^0 = v^0_{R,d_1,d_2}$. Similarly we have

$$r_{L,d_1,d_2}(t) > -d_1 + \left(aM' - \frac{c'}{M'} \right) t \quad \text{for } t < T(M', v^0_{L,d_1,d_2}), \tag{3.12}$$

where $r_{L,d_1,d_2}(t)$ is the right interface curve of the solution $v_{L,d_1,d_2}(t,x)$ with initial value $v_{L,d_1,d_2}(0,x) = v^0_{L,d_1,d_2}(x) \equiv \psi(-x;\varepsilon,d_1,d_2)$, and $r_{L,d_1,d_2}(0) = -d_1$. Since it follows from the definition of the function $\psi(x;\varepsilon,d_1,d_2)$ that

$$\begin{cases} C_j(v^0_{R,d_1,d_2}) = C_j(\phi), \ C_j(v^0_{L,d_1,d_2}) = C_j(\phi), \\ C_j(v^0(\cdot\,;\varepsilon,d_1,d_2)) = C_j(\phi) \ (j = 0,1,2) \quad \text{for all } d_1, d_2 > 0, \end{cases} \tag{3.13}$$

we see that $T(M', v^0_{L,d_1,d_2})$ and $T(M', v^0_{R,d_1,d_2})$ are also independent of $d_j (j = 1,2)$. From (3.7) we can choose a positive constant T' satisfying

$$T' < \min\Big\{ T(M', v^0_{L,d_1,d_2}),\ T(M', v^0_{R,d_1,d_2}) \Big\}, \tag{3.14}$$

$$\frac{(\phi(\beta;\varepsilon))^{m-1} - (c' + mC_0(\phi)C_2(\phi))T'}{c' + mC_0(\phi)C_2(\phi)} > \frac{\varepsilon^{m-1}}{c'}. \tag{3.15}$$

We fix T', and choose a positive constant $d_1 < \left(aM' - \frac{c'}{M'} \right) T'$. Then we have from (3.11) and (3.12)

$$\ell_{R,d_1,d_2}(T') < 0 \ \text{ and } \ r_{L,d_1,d_2}(T') > 0,$$

which implies that $\ell(t') = r(t')$ holds for some $t' < T'$. Since the number of the local maximum points of $v(t,\cdot\,;\varepsilon,d_1,d_2)$ never exceeds *two*, supp $v(t,\cdot\,;\varepsilon,d_1,d_2)$ becomes connected for each $t \in (t', T']$ by the comparison theorem which is concerned with the initial data([2]).

We next take an arbitrary constant $T_1 (t' < T_1 < T')$, and show that $v(T_1, x; \varepsilon, d_1, d_2)$ instead of v^0 satisfies (2.25) for sufficiently large d_2. By Lemma 2.3, (2.1), (3.15) and Condition C we obtain

$$\begin{aligned} u(t, \pm(d_1+d_2+\beta);\varepsilon,d_1,d_2) &\geq u^0(\pm(d_1+d_2+\beta);\varepsilon,d_1,d_2)\\ &\quad - \left\{ c' + mC_0(v^0(\cdot\,;\varepsilon,d_1,d_2))C_2(v^0(\cdot\,;\varepsilon,d_1,d_2)) \right\} t\\ &= (\phi(\beta;\varepsilon))^{m-1} - \{c' + mC_0(\phi)C_2(\phi)\}\, t > 0 \quad \text{for } t \leq T', \end{aligned} \tag{3.16}$$

and

$$\frac{u(T_1,\pm(d_1+d_2+\beta);\varepsilon,d_1,d_2)}{c'+mC_0(v(T_1,\cdot\,;\varepsilon,d_1,d_2))C_2(v(T_1,\cdot\,;\varepsilon,d_1,d_2))} \geq \frac{(\phi(\beta;\varepsilon))^{m-1}-\{c'+mC_0(\phi)C_2(\phi)\}\,T_1}{c'+mC_0(\phi)C_2(\phi)} > \frac{\varepsilon^{m-1}}{c'}. \tag{3.17}$$

Since $v(T_1,x;\varepsilon,d_1,d_2)>0$ on $[-d_1-d_2-\beta,\ d_1+d_2+\beta]$, we have from Lemma 2.3 and (2.1),

$$\begin{aligned}\|u(T_1,\cdot\,;\varepsilon,d_1,d_2)\|_{L^1(\mathbf{J}_{d_1,d_2})} &\leq \|u^0(\cdot\,;\varepsilon,d_1,d_2)\|_{L^1(\mathbf{J}_{d_1,d_2})}\\ &\quad+(m+a)\|u^0(\cdot\,;\varepsilon,d_1,d_2)\|_\infty TV\left(u^0_x(\cdot\,;\varepsilon,d_1,d_2)\right)T_1,\end{aligned} \tag{3.18}$$

where $\mathbf{J}_{d_1,d_2}=[-\gamma-d_1-d_2,\ \gamma+d_1+d_2]$. Then, for sufficiently large d_2, we have

$$\begin{aligned}0&<\frac{\|u(T_1,\cdot\,;\varepsilon,d_1,d_2)\|_{L^1(\mathbf{J}_{d_1,d_2})}}{2c'(\gamma+d_1+d_2)-(m+a)C_0(v(T_1,\cdot\,;\varepsilon,d_1,d_2))TV\left(u_x(T_1,\cdot\,;\varepsilon,d_1,d_2)\right)}\\ &\leq \frac{\varepsilon^{m-1}+\dfrac{(m+a)C_0(\phi)TV\left((\phi^{m-1})_x\right)T_1}{\gamma+d_1+d_2}}{c'-\dfrac{(m+a)C_0(\phi)TV\left((\phi^{m-1})_x\right)}{\gamma+d_1+d_2}},\end{aligned} \tag{3.19}$$

and observe that the the right hand side of (3.19) converges to $\dfrac{\varepsilon^{m-1}}{c'}$ as $d_2\to\infty$. From (3.17) we can choose d_2 sufficiently large so that

$$\begin{aligned}&\frac{u(T_1,\pm(d_1+d_2+\beta_j);\varepsilon,d_1,d_2)}{c'+mC_0(v(T_1,\cdot\,;\varepsilon,d_1,d_2))C_2(v(T_1,\cdot\,;\varepsilon,d_1,d_2))}\\ &\quad>\frac{\|u(T_1,\cdot\,;\varepsilon,d_1,d_2)\|_{L^1(\mathbf{J}_{d_1,d_2})}}{2c'(\gamma+d_1+d_2)-(m+a)C_0(v(T_1,\cdot\,;\varepsilon,d_1,d_2))TV\left(u_x(T_1,\cdot\,;\varepsilon,d_1,d_2)\right)}>0.\end{aligned} \tag{3.20}$$

Thus (2.25) is satisfied by putting $\beta_1=-\beta-d_1-d_2$, $\beta_2=\beta+d_1+d_2$, $\gamma_1=-\gamma-d_1-d_2$ and $\gamma_2=\gamma+d_1+d_2$. Hence there exist $T_2(>T_1)$ and $\tilde{x}\in[\gamma_1,\gamma_2]$ satisfying (3.8), and the proof is complete.

3.3. Proof of Main Theorem. The constants ε, d_1, d_2 are given in Theorem 3.1 and fixed. So, for simplicity we put $v_\rho(t,x)=v_\rho(t,x;\varepsilon,d_1,d_2)$ and $u_\rho(t,x)=u_\rho(t,x;\varepsilon,d_1,d_2)\equiv(v_\rho(t,x;\varepsilon,d_1,d_2))^{m-1}$.

We first note that $v_\rho(t,\pm(d_1+d_2+\beta))>0$ for $t<T_1$ and $\rho>0$ (see (3.16)). Putting $\mathbf{S}=[0,\ T_1]\times[-d_1,d_1]$, we show that $\mathbf{S}$ contains at least one point $(\tilde{t},\hat{x})$ such that $v_{\tilde{\rho}}(\tilde{t},\hat{x})=0$ for some positive constant $\tilde{\rho}$. For this end we assume the contrary; that is, suppose $v_\rho(t,x)>0$ on $\mathbf{S}$ for $\rho>0$. By Lemma 2.3, Condition D and (2.1) we obtain

$$\begin{aligned}\int_{-d_1}^{d_1}u_\rho(t,x)dx&=\int_{-d_1}^{d_1}u_\rho(0,x)dx\\ &\quad+\int_0^t\int_{-d_1}^{d_1}\left\{mu_\rho(t,x)u_{\rho xx}(t,x)+a(u_{\rho x}(t,x))^2-c'\right\}dxdt\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&= 2d_1\rho^{m-1} \\
&- \int_0^t \left\{ 2d_1 c' - (m-2)a \int_{-d_1}^{d_1} u_\rho(t,x) u_{\rho xx}(t,x)dx - a\Big[u_\rho(t,x)u_{\rho x}(t,x)\Big]_{-d_1}^{d_1} \right\} dt \\
&\le 2d_1\rho^{m-1} - \left\{ 2d_1 c' - a \max_{[0,t]\times[-d_1,d_1]} u_\rho(t,x)\Big((2-m)TV(u^0_{\rho x}) + 2\|u^0_{\rho x}\|_\infty\Big)\right\} t \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \in [0, T_1].
\end{aligned}
$$

Let ρ_1 be an arbitrary fixed positive constant such that

$$
\rho_1{}^{m-1} < \frac{2d_1 c'}{a\Big((2-m)TV(u^0_x) + 2\|u^0_x\|_\infty\Big)}. \tag{3.22}
$$

Then, by the continuity of the solution $v_\rho(t,x)$ and the comparison theorem on the initial data([2]) there exist positive constants ρ_2 and $\tilde{T}_1 < T_1$ such that

$$
\max_{[0,t]\times[-d_1,d_1]} u_\rho(t,x) < \rho_1{}^{m-1} \tag{3.23}
$$
$$
\text{for} \quad t < \tilde{T}_1 \quad \text{and} \quad \rho < \rho_2 < \min\left(\rho_1, \psi(d_1+\eta, \varepsilon, d_1, d_2)\right).
$$

We put

$$
T(\rho) = \frac{2d_1\rho^{m-1}}{2d_1 c' - a\rho_1{}^{m-1}\Big((2-m)TV(u^0_x) + 2\|u^0_x\|_\infty\Big)}, \tag{3.24}
$$

and choose $\tilde{\rho} < \rho_2$ such that $T(\tilde{\rho}) < \tilde{T}_1$. Hence, it follows from (3.21) and Condition D that

$$
\int_{-d_1}^{d_1} u_{\tilde{\rho}}(t,x)dx < 0 \quad \text{for} \quad t \in (T(\tilde{\rho}), \tilde{T}_1], \tag{3.25}
$$

which is a contradiction. Thus, $v_{\tilde{\rho}}(T_0, \hat{x}) = 0$ holds for some $(T_0, \hat{x}) \in \mathbf{S}$. It is clear by Theorem 3.1 and the comparison theorem that supp $v_{\tilde{\rho}}(T_1, \cdot)$ becomes connected. Since $u_{\tilde{\rho}}(0,x) = u^0_{\tilde{\rho}}(x) < \varepsilon^{m-1}$ holds on $[-d_1, d_1]$, the inequalities (3.19) and (3.20) also hold with $u = u_{\tilde{\rho}}$. Thus (3.10) follows and the proof is complete.

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