Hierarchical structure of Green functions and the best constant of Sobolev inequality corresponding to a bending problem of a beam

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We consider 16 different 2 points boundary value problems for a bending of a beam.

BVP (α, β)

$$\begin{cases} u^{(4)} \,-\, p\,u'' \,+\, q\,u \,=\, f(x) & (-1 < x < 1) \\ u_{\alpha_i}(-1) \,=\, u_{\beta_i}(1) \,=\, 0 & (i = 0, 1) \end{cases}$$

$u(x)$: bending of a beam, $f(x)$: density of a load, $p > 0$: tension, $q > 0$: spring constant. We assume that $(p/2)^2 > q > 0,\ p > 0$. This assumption is equivalent to $p = a^2 + b^2,\ q = a^2 b^2,\ a > b > 0$. $u_0 = u,\ u_1 = u',\ u_2 = u'',\ u_3 = u''',\ u_{\widetilde{3}} = u_3 - p u_1$. $\alpha = (\alpha_0, \alpha_1)$ and $\beta = (\beta_0, \beta_1)$ takes values $(0, 1)$ (clamped), $(0, 2)$ (Dirichlet), $(1, 3)$ (Neumann) and $(2, \widetilde{3})$ (free).

For any bounded continuous function $f(x)$, BVP(α, β) has a unique classical solution

$$u(x) \,=\, \int_{-1}^{1} G(x, y)\, f(y)\, dy \qquad (-1 < x < 1)$$

where $G(x, y) = G(\alpha, \beta; x, y)$ is Green function.

Theorem 1 *All Green functions are positive valued and have the suitable hierarchical structure of 16 Green functions.*

We introduce Sobolev space $H = \left\{ \ u(x) \ \middle| \ u(x),\ u'(x),\ u''(x) \in L^2(-1, 1), \quad A(\alpha, \beta) \ \right\}$

$$\begin{aligned} A(0,1,0,1) &: \quad u(-1) = u'(-1) = u(1) = u'(1) = 0 \\ A(0,2,0,2) &: \quad u(-1) = u(1) = 0 \\ A(1,3,1,3) &: \quad u'(-1) = u'(1) = 0 \\ A(2,\widetilde{3},2,\widetilde{3}) &: \quad \textit{none} \end{aligned}$$

and Sobolev energy $\|\, u \,\|_H^2 \,=\, \displaystyle\int_{-1}^{1} \left[\ |\, u''(x)\,|^2 + p\,|\, u'(x)\,|^2 + q\,|\, u(x)\,|^2 \ \right] dx.$

Theorem 2 *For any function $u(x) \in H$, there exists a positive constant C which is independent of $u(x)$ such that Sobolev inequality*

$$\left(\sup_{|y| \le 1} |\, u(y)\,| \right)^2 \,\le\, C\, \|\, u \,\|_H^2$$

holds. Among such C the best constant $C(\alpha, \beta)$ is given by

$$C(\alpha, \beta) \,=\, \max_{|y| \le 1} G(y, y) \,=\, G(y_0, y_0).$$

If we replace C by $C(\alpha, \beta)$ in the above inequality, then the equality holds for $u(x) = c\,G(x, y_0)$ $(-1 < x < 1)$ where c is any complex number.