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Exponential decay of the principal eigenvalue of an elliptic problem with large drift [1]

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1. Eigenvalue problem with large drift

Let Ω be bounded Lipschitz domain in $\mathbf{R}^n$ $(n \geq 1)$ and let $p \in \mathbf{R}$. For a given vector field $\boldsymbol{a} \in W^{1,\infty}(\Omega, \mathbf{R}^n)$, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta v(\boldsymbol{x}) + p\boldsymbol{a}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) = \lambda v(\boldsymbol{x}) & (\boldsymbol{x} \in \Omega) \\ v(\boldsymbol{x}) = 0 & (\boldsymbol{x} \in \partial\Omega) \\ v(\boldsymbol{x}) > 0 & (\boldsymbol{x} \in \Omega) \end{cases} \tag{1}$$

$\lambda_1(p) > 0$: the principal eigenvalue, $\qquad v_1(\boldsymbol{x}, p)$ $(\boldsymbol{x} \in \Omega)$: the principal eigenfunction

Our interests: Behaviours of $\lambda_1(p)$ as $p \to \infty$. $\lambda_1(p) \longrightarrow 0,\ \bar{\lambda},\ \infty$?
1. Classification by the flow profile $\boldsymbol{a}(\boldsymbol{x})$.
2. Precise asymptotic behaviour in case of "exponetial decay phenomenon".

$$0 < \lambda_1(p) \leq Ce^{-bp} \quad \text{as } p \geq p_0 \qquad \text{(where } C,\ b,\ p_0 \text{ are positive constants)}$$

2. Selfadjoint assumption and Rayleigh quotient

For $\boldsymbol{a}(\boldsymbol{x})$, we assume that its velocity potential exists:

$$\textbf{Selfadjoint assumption: } {}^\exists b \in W^{2,\infty}(\Omega) \quad \text{s.t.} \quad \boldsymbol{a} = \nabla b.$$

Then the eigenvalue problem (1) is equivalent to

$$\begin{cases} -\Delta w(\boldsymbol{x}) + q(\boldsymbol{x}, p) w(\boldsymbol{x}) = \lambda w(\boldsymbol{x}) & (\boldsymbol{x} \in \Omega) \\ w(\boldsymbol{x}) = 0 & (\boldsymbol{x} \in \Gamma) \end{cases} \tag{2}$$

where

$$w(\boldsymbol{x}) := e^{-\frac{p}{2} b(\boldsymbol{x})} v(\boldsymbol{x}), \qquad q(\boldsymbol{x}, p) := -\frac{p}{2} \mathrm{div} \boldsymbol{a}(\boldsymbol{x}) + \frac{p^2}{4} |\boldsymbol{a}(\boldsymbol{x})|^2.$$

The principal eigenvalue $\lambda_1(p)$ is given by the Rayleigh quotient:

$$\lambda_1(p) = \min_{w \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla w(\boldsymbol{x})|^2 + q(\boldsymbol{x}, p) w(\boldsymbol{x})^2) d\boldsymbol{x}}{\int_\Omega w(\boldsymbol{x})^2 d\boldsymbol{x}} = \min_{v \in H_0^1(\Omega)} \frac{\int_\Omega e^{-pb(\boldsymbol{x})} |\nabla v(\boldsymbol{x})|^2 d\boldsymbol{x}}{\int_\Omega e^{-pb(\boldsymbol{x})} v(\boldsymbol{x})^2 d\boldsymbol{x}}.$$

3. Our results in n-dimensional case

A comparison theorem for eigenvalues derived from the Rayleigh quotient formula gives the following theorem.

Theorem 1. *Let $\lambda_\Omega > 0$ be the first eigenvalue of $-\Delta_D$.*
(1) If $\boldsymbol{a} \not\equiv 0$, $0 <^\forall C_1 < 1/4 <^\forall C_2$, ${}^\exists p_0 > 0$ s.t.

$$C_1 \left(\inf_{\boldsymbol{x} \in \Omega} |\boldsymbol{a}(\boldsymbol{x})|^2 \right) p^2 \leq \lambda_1(p) \leq C_2 \left(\sup_{\boldsymbol{x} \in \Omega} |\boldsymbol{a}(\boldsymbol{x})|^2 \right) p^2 \qquad ({}^\forall p \geq p_0).$$

(2) $\lambda_1(p) \geq \lambda_\Omega + \left(-\operatorname*{ess\text{-}sup}_{\boldsymbol{x} \in \Omega} \mathrm{div} \boldsymbol{a}(\boldsymbol{x}) \right) \dfrac{p}{2} \qquad (p > 0).$

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Corollary No exponential decay phenomenon if $\inf|\boldsymbol{a}(\boldsymbol{x})|>0$ or ess-sup $\mathrm{div}\boldsymbol{a}(\boldsymbol{x})\le 0$.

Potential well condition : $^{\exists}\Omega'$: a Lipschitz subdomain of Ω, s.t. $\min_{\boldsymbol{x}\in\overline{\Omega'}} b(\boldsymbol{x}) < \min_{\boldsymbol{x}\in\partial\Omega'} b(\boldsymbol{x})$.

Under the potential well condition, we define $b_1 := \min_{\boldsymbol{x}\in\overline{\Omega'}} b(\boldsymbol{x})$ and $b_2 := \min_{\boldsymbol{x}\in\partial\Omega'} b(\boldsymbol{x})$. Then $b_2 - b_1 > 0$ represents the depth of the potential well Ω'.

Theorem 2. *Under the potential well condition,* $^{\forall}\sigma\in(0,b_2-b_1)$, $^{\exists}C>0$ *s.t.*

$$0<\lambda_1(p)\le Ce^{-\sigma p} \qquad (p\ge 0).$$

Theorem 3. *Let* $\Omega_j\subset\Omega$ $(j=1,\cdots,m)$ *be the potential wells with* $\overline{\Omega_i}\cap\overline{\Omega_j}=\emptyset$ $(i\ne j)$ *and let* σ_0 *be the minimum depth of the potential well among* Ω_j $(j=1,\cdots,m)$. *Then* $^{\forall}\sigma\in(0,\sigma_0)$, $^{\exists}C>0$ *s.t.*

$$0<\lambda_1(p)<\lambda_2(p)\le\cdots\le\lambda_m(p)\le Ce^{-\sigma p} \qquad (p\ge 0).$$

4. Precise asymptotic behavior in 1-dimensional case

$$\begin{cases} -v''(x)+pa(x)v'(x)=\lambda v(x) & (-1<x<1) \\ \quad v(-1)=v(1)=0, & v(x)>0 \quad (-1<x<1) \end{cases}$$

Assumptions

$a(x)=b'(x)$ with $b\in W^{1,\infty}(-1,1)$: even, and $\max_{x\in[0,1]}\left\{x \,\middle|\, b(x)=\min_{0\le s\le 1} b(s)\right\} < \min_{x\in[0,1]}\left\{x \,\middle|\, b(x)=\max_{0\le s\le 1} b(s)\right\}$.

Example : $a(x)=x, \quad b(x)=\frac{1}{2}x^2 \quad\Rightarrow\quad 0<1.$

Theorem 4. *Under the above assumptions, we have*

$$\lambda_1(p)\sim\left(\int_0^1 e^{-pb(x)}dx\right)^{-1}\left(\int_0^1 e^{pb(y)}dy\right)^{-1} \qquad (as\ p\to\infty)$$

Corollary $b_0:=\max_{0\le x\le 1} b(x)-\min_{0\le x\le 1} b(x)$, $^{\forall}\varepsilon\in(0,b_0)$, $^{\exists}M_1,M_2,p_0>0$ s.t.

$$M_1e^{-p(b_0+\varepsilon)}\le\lambda_1(p)\le M_2e^{-p(b_0-\varepsilon)} \quad (\forall p\ge p_0) \ \text{ i.e. } \lim_{p\to\infty}\left(-\frac{1}{p}\log\lambda_1(p)\right)=b_0$$

Theorem 5. *If* $a(x)=x$ *(i.e.* $-v''(x)+pxv'(x)=\lambda v(x)$ $(-1<x<1)$ *), then*

$$\lambda_1(p)\sim\sqrt{\frac{2}{\pi}}p^{\frac{3}{2}}e^{-\frac{1}{2}p} \quad (p\to\infty).$$

References

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