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NUMERICAL SOLUTION FOR THE WILLMORE FLOW OF GRAPHS

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Abstract. In this article we present a numerical scheme for the Willmore flow of graphs. It is based on the method of lines. Resulting ordinary differential equations are solved using the 4th order Runge-Kutta-Merson solver. We show basic properties of the semi-discrete scheme and present several computational studies of evolving graphs.

Key words. Willmore flow, method of lines, curvature minimization, gradient flow, Laplace-Beltrami operator, Gauss curvature

AMS subject classifications. 35K35, 35K55, 53C44, 65M12, 65M20, 74S20

1. Introduction. For the purpose of this article we consider evolution of two dimensional surface $\Gamma(t)$ embedded in $\mathbb{R}^3$ such that it can be described as a graph of some function $u : (0,T\rangle \times \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$. We investigate the following law

$$V = 2\triangle_\Gamma H + H^3 - 4HK \text{ on } \Gamma(t), \tag{1.1}$$

where V is the normal velocity, $\triangle_\Gamma$ is the Laplace-Beltrami operator, $H = \kappa_1 + \kappa_2$ is the mean curvature, $K = \kappa_1 \cdot \kappa_2$ is the Gauss curvature and κ_1 and κ_2 denote the principal curvatures of the surface.

As follows from [5, 6, 7] the law (1.1) represents the L_2-gradient flow for the functional W defined as:

$$W(f) = \int_\Gamma H^2 dS, \quad \Gamma = \{(\mathbf{x}, u(x)) \mid \mathbf{x} \in \Omega\}. \tag{1.2}$$

The gradient flow approach is described e.g. in [13]. Existence of the solution under certain initial conditions was proved in [12, 8]. In [5] an implicit numerical scheme for the Willmore flow of graphs based on the finite element method together with the numerical analysis is presented. A level set formulation for the Willmore flow can be found in [6]. For the physical meaning of the minimization of (1.2) we refer to [4]. In [7] the authors describe an algorithm for evolution of elastic curves in $\mathbb{R}^n$. An interesting algorithm for parametrised curves driven by intrinsic Laplacian of curvature can be found in [9] where the authors use the tangential vector for redistribution of the control points on the curve. Application for the surface reconstruction of scratched objects is discused in [14].

We present a numerical scheme for the Willmore flow of graphs based on the method of lines. For discretization in time we use the 4th order Runge-Kutta type solver having explicit nature. This method was succesfully used for solving several problems in interface motion [2]. Our work is also related to [3] where the surface diffusion for graphs is treated by a similar approach.

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2. Problem formulation. We assume that $\Gamma(t)$ is a graph of a function u of two variables:

$$\Gamma(t) = \left\{[\mathbf{x}, u(t, \mathbf{x})] \mid \mathbf{x} \in \Omega \subset \mathbb{R}^2\right\},$$

where $\Omega \equiv (0, L_1) \times (0, L_2)$ is an open rectangle, $\partial\Omega$ its boundary and ν its outer normal.

We express the quantities of (1.1) in terms of the graph description of $\Gamma(t)$ see [3]:

$$Q = \sqrt{1 + |\nabla u|^2}; \quad \mathbf{n} = \frac{(\nabla u, -1)}{Q}, \tag{2.1}$$

$$V = -\frac{u_t}{Q}, \tag{2.2}$$

$$H = \nabla \cdot \mathbf{n}, \tag{2.3}$$

$$K = \frac{\det D^2 u}{Q^4}, \tag{2.4}$$

$$\triangle_\Gamma H = \frac{1}{Q} \nabla \cdot \left[\left(QI - \frac{\nabla u \otimes \nabla u}{Q}\right) \nabla H\right]. \tag{2.5}$$

Lemma 2.1. *For the graph formulation of the Willmore flow, (1.1) takes the following form*

$$\frac{\partial u}{\partial t} = -Q\nabla \cdot \left[\frac{2}{Q}(\mathbb{I} - \mathbb{P}) \nabla w - \frac{w^2}{Q^3} \nabla u\right], \tag{2.6}$$

$$w = Q\nabla \cdot \frac{\nabla u}{Q}, \tag{2.7}$$

where

$$\mathbb{P} = \frac{\nabla u}{Q} \otimes \frac{\nabla u}{Q}, \quad (u \otimes v)_{ij} = u_i \cdot v_j.$$

Proof. The proof follows [5]. It is given here because of better understanding of consequent results. We start with the expression (2.5) which can be written as

$$\triangle_\Gamma H = \nabla \cdot \left(\frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right) \nabla (QH)\right) - H\nabla \cdot \left(\frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right) \nabla Q\right). \tag{2.8}$$

Using (2.3) we have

$$\frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right) \nabla Q = \frac{1}{Q}\left(\nabla Q - \frac{\triangle u}{Q} \nabla u\right) + H\frac{\nabla u}{Q}, \tag{2.9}$$

from (2.4) and by a brief rearangement we obtain

$$\nabla \cdot \left(\frac{1}{Q}\left(\nabla Q - \frac{\triangle u}{Q} \nabla u\right)\right) = -2K. \tag{2.10}$$

Putting (2.9) and (2.10) into (2.8) we have

$$\begin{aligned} \triangle_\Gamma H &= \nabla \cdot \left(\frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right) \nabla (QH)\right) + 2HK - H\nabla \cdot \left(H\frac{\nabla u}{Q}\right) \\ &= \nabla \cdot \left(\frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right) \nabla (QH)\right) + 2HK - \frac{1}{2}\nabla \cdot \left(\frac{H^2}{Q} \nabla u\right) - \frac{1}{2}H^3. \end{aligned}$$

Together with (1.1), (2.2) and (2.7) we obtain (2.6). □

The above lemma allows to introduce the following problem:

Definition 2.2. *The graph formulation for the Willmore flow is a system of two partial differential equations of the second order for u and w in the form*

$$\frac{\partial u}{\partial t} = -Q\nabla\cdot\left[\frac{2}{Q}\left(\mathbb{I}-\mathbb{P}\right)\nabla w - \frac{w^2}{Q^3}\nabla u\right] \text{ in } \Omega\times(0,T), \tag{2.11}$$

$$w = Q\nabla\cdot\frac{\nabla u}{Q}, \tag{2.12}$$

$$u(\cdot,0) = u_{ini},$$

with the Dirichlet boundary conditions

$$u\,|_{\partial\Omega} = 0, \quad w\,|_{\partial\Omega} = 0, \tag{2.13}$$

or with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}\,|_{\partial\Omega} = 0, \quad \frac{\partial w}{\partial \nu}\,|_{\partial\Omega} = 0. \tag{2.14}$$

Remark: Multiplying (2.11) by test function $\varphi \in H_0^1(\Omega)$ in the case of the Dirichlet boundary conditions, or $\varphi \in H^1(\Omega)$ for the Neumann boundary conditions, summing over Ω and applying the Green theorem we have

$$\begin{aligned}\int_\Omega \frac{u_t}{Q}\varphi &= -\int_\Omega \nabla\cdot\left[\frac{2}{Q}\left(\mathbb{I}-\mathbb{P}\right)\nabla w - \frac{w^2}{Q^3}\nabla u\right]\varphi \\ &= \int_\Omega\left[\frac{2}{Q}\left(\mathbb{I}-\mathbb{P}\right)\nabla w - \frac{w^2}{Q^3}\nabla u\right]\cdot\nabla\varphi - \\ &\quad \int_{\partial\Omega}\left[\left(\frac{2}{Q}\left(\mathbb{I}-\mathbb{P}\right)\nabla w\right)\cdot\nu - \frac{w^2}{Q^3}\frac{\partial w}{\partial \nu}\right]\varphi.\end{aligned}$$

The last term vanishes because of the choice of the test function φ in the case of the Dirichlet boundary conditions (2.13). In the case of the Neumann boundary conditions the sum over $\partial\Omega$ vanishes because of (2.14) and the fact that

$$\left(\left(\mathbb{I}-\mathbb{P}\right)\cdot\nabla w\right)\nu = \frac{\partial w}{\partial \nu} - \frac{1}{Q^2}\left(\left(\nabla u\otimes\nabla u\right)\nabla w\right)\cdot\nu = \frac{\partial w}{\partial \nu} - \frac{\nabla u\cdot\nabla w}{Q^2}\frac{\partial u}{\partial \nu}.$$

Similarly we multiply (2.12) by test function $\xi \in H_0^1(\Omega)$ for the Dirichlet boundary conditions resp. $\xi \in H^1(\Omega)$ for the Neumann boundary conditions and we have

$$\int_\Omega \frac{w}{Q}\xi = \int_\Omega\left(\nabla\cdot\frac{\nabla u}{Q}\right)\xi = -\int_\Omega\frac{\nabla u\cdot\nabla\xi}{Q} + \int_{\partial\Omega}\frac{\xi}{Q}\frac{\partial u}{\partial \nu}.$$

The last term vanishes because of the choice of ξ in the case of the Dirichlet boundary conditions or because of (2.14) in the case of the Neumann boundary conditions.

We can define the weak solution for the Willmore flow of graphs as follows:

Definition 2.3. *The weak solution of the graph formulation for the Willmore flow with homogeneous Dirichlet boundary conditions is a couple $u, w : (0,T) \to H_0^1(\Omega)$ which satisfy a.e in $(0,T)$, for each test functions $\varphi, \xi \in H_0^1(\Omega)$*

$$\int_\Omega \frac{u_t}{Q}\varphi = \int_\Omega \frac{2}{Q}\left[\left(\mathbb{I}-\mathbb{P}\right)\nabla w\right]\cdot\nabla\varphi - \int_\Omega\frac{w^2}{Q^3}\nabla u\cdot\nabla\varphi \text{ a.e. in } (0,T) \tag{2.15}$$

$$\int_\Omega \frac{w}{Q}\xi = -\int_\Omega\frac{\nabla u\cdot\nabla\xi}{Q}. \tag{2.16}$$

with the initial condition

$$u\mid_{t=0}=u_{ini}.$$

Weak solution for the problem with homogeneous Neumann boundary conditions is a couple $u, w : (0,T) \to H^1(\Omega)$ *which satisfy (2.15) a.e. in* $(0,T)$*, for each test functions* $\varphi, \xi \in H^1(\Omega)$*.*

Remark: There are at least two different steady solutions for the Willmore flow of graphs. The trivial solution is represented by a constant function u(specified by the boundary conditions) and zero mean curvature ($w = 0$) The second solution is induced by a sphere with given radius r since the principal curvatures are $\kappa_1 = \kappa_2 = \frac{1}{r}$ and so $H = \kappa_1 + \kappa_2 = \frac{2}{r}$ and $K = \kappa_1\kappa_2 = \frac{1}{r^2}$. From this fact it follows that the right hand side of (1.1) is equal to zero. In this case, the boundary conditions are different from (2.13) and (2.14).

Mathematical properties of (1.1) have been partially studied in [12] for the case when the initial condition is close to a sphere and in [8] existence of the solution was proved under the assumption that $\int_\Gamma |A^\circ|^2$ is sufficiently small, for A° denoting the trace-free part of the second fundamental form.

3. Numerical scheme. For the numerical solution of (1.1), we will use method of lines with finite difference discretization in space.

We use the following notation. Let h_1, h_2 be space steps such that $h_1 = \frac{L_1}{N_1}$ and $h_2 = \frac{L_2}{N_2}$ for some $N_1, N_2 \in \mathbb{N}^+$. We define a uniform grid as

$$\omega_h = \{(ih_1, jh_2) \mid i = 1 \cdots N_1 - 1, j = 1 \cdots N_2 - 1\},$$
$$\overline{\omega}_h = \{(ih_1, jh_2) \mid i = 0 \cdots N_1, j = 0 \cdots N_2\}.$$

For $u : \mathbb{R}^2 \to \mathbb{R}$ we define a projection on $\overline{\omega}_h$ as $u_{ij} = u(ih_1, jh_2)$. We introduce the differences in agree with [11] as follows:

$$u_{\overline{x}_1,ij} = \frac{u_{ij} - u_{i-1,j}}{h_1}, \quad u_{x_1,ij} = \frac{u_{i+1,j} - u_{ij}}{h_1},$$
$$u_{\overline{x}_2,ij} = \frac{u_{ij} - u_{i,j-1}}{h_2}, \quad u_{x_2,ij} = \frac{u_{i,j+1} - u_{ij}}{h_2},$$
$$\overline{\nabla}_h u_{ij} = (u_{\overline{x}_1,ij}, u_{\overline{x}_2,ij}), \nabla_h u_{ij} = (u_{x_1,ij}, u_{x_2,ij}).$$

The discrete operator for divergence is defined in the same manner. For $f, g : \overline{\omega}_h \to \mathbb{R}$ and $\boldsymbol{f}, \boldsymbol{g} : \overline{\omega}_h \to \mathbb{R}^2$ we define

$$(f,g)_h = \sum_{i,j=1}^{N_1-1,N_2-1} h_1 h_2 f_{ij} g_{ij} \quad , \quad \|f\|_h^2 = (f,f)_h,$$
$$\left(\boldsymbol{f}^1, \boldsymbol{g}^1\right\rfloor = \sum_{i,j=1}^{N_1,N_2-1} h_1 h_2 \boldsymbol{f}^1_{ij} \boldsymbol{g}^1_{ij} \quad , \quad \left(\boldsymbol{f}^2, \boldsymbol{g}^2\right\rceil = \sum_{i,j=1}^{N_1-1,N_2} h_1 h_2 \boldsymbol{f}^2_{ij} \boldsymbol{g}^2_{ij},$$
$$(\boldsymbol{f}, \boldsymbol{g}] = \left(\boldsymbol{f}^1, \boldsymbol{g}^1\right\rfloor + \left(\boldsymbol{f}^2, \boldsymbol{g}^2\right\rceil \quad , \quad \|\boldsymbol{f}]^2 = (\boldsymbol{f}, \boldsymbol{f}].$$

For the discretization of the Neumann boundary conditions we define the grid boundary normal difference $u_{\bar{n}}$:

$$\begin{aligned} u_{\bar{n},0j} &= u_{\bar{x}_1,1j} && \text{for } j = 0, \dots, N_2, \\ u_{\bar{n},N_1j} &= u_{\bar{x}_1,N_1j} && \text{for } j = 0, \dots, N_2, \\ u_{\bar{n},i0} &= u_{\bar{x}_2,i1} && \text{for } i = 0, \dots, N_1, \\ u_{\bar{n},iN_2} &= u_{\bar{x}_2,iN_2} && \text{for } i = 0, \dots, N_1. \end{aligned}$$

For the purpose of analysis, we recall the grid version of the Green formula proved in [1]:

LEMMA 3.1. *Let $p, u, v : \bar{\omega}_h \to \mathbb{R}$. Then the Green formula is valid:*

$$(\nabla_h \cdot (p\overline{\nabla}_h u), v)_h = -(p\overline{\nabla}_h u, \overline{\nabla}_h v] + \sum_{j=1}^{N_2-1} h_2 (pu_{\overline{x}_1}|_{N_1 j} v_{N_1 j} - pu_{\overline{x}_1}|_{1j} v_{0j}) + \sum_{i=1}^{N_1-1} h_1 (pu_{\overline{x}_2}|_{iN_2} v_{iN_2} - pu_{\overline{x}_2}|_{i1} v_{i0}). \tag{3.1}$$

If we denote

$$\overline{Q}^h_{ij} = \sqrt{1 + \frac{1}{2}\left(u^2_{\overline{x}_1,ij} + u^2_{x_1,ij} + u^2_{\overline{x}_2,ij} + u^2_{x_2,ij}\right)},$$
$$i = 1, \cdots, N_1 - 1, \quad j = 1, \cdots, N_2 - 1,$$
$$Q^h_{ij} = \sqrt{1 + u^2_{\overline{x}_1,ij} + u^2_{\overline{x}_2,ij}},$$
$$i = 1, \cdots, N_1, \quad j = 1, \cdots, N_2,$$
$$\mathbb{E}^h_{ij} = \frac{2}{Q_{ij}} \begin{pmatrix} 1 - u^2_{\overline{x}_1,ij} & -u_{\overline{x}_1,ij} u_{\overline{x}_2,ij}, \\ -u_{\overline{x}_1,ij} u_{\overline{x}_2,ij} & 1 - u^2_{\overline{x}_2,ij} \end{pmatrix},$$
$$i = 1, \cdots, N_1, \quad j = 1, \cdots, N_2.$$

then the scheme has the following form

$$\frac{du^h}{dt} = -\overline{Q}^h \nabla_h \left(\frac{1}{Q^h} \mathbb{E}^h \overline{\nabla}_h w^h - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3} \overline{\nabla}_h u^h \right), \tag{3.2}$$

$$w^h = Q^h \cdot \left[\left(\frac{{u^h}_{\overline{x}_1}}{Q^h} \right)_{x_1} + \left(\frac{{u^h}_{\overline{x}_2}}{Q^h} \right)_{x_2} \right], \tag{3.3}$$

(3.4)

and the initial condition is

$$u^h(0) = u_{ini}|_{\overline{\omega}_h}.$$

We consider either the Dirichlet boundary conditions

$$u^h|_{\partial\omega_h} = 0, \quad w^h|_{\partial\omega_h} = 0, \tag{3.5}$$

or the Neumann boundary conditions

$$u^h_{\bar{n}}|_{\partial\omega_h} = 0, \quad w^h_{\bar{n}}|_{\partial\omega} = 0. \tag{3.6}$$

The following theorem shows the energy equality of the scheme.

THEOREM 3.2. *For $u^h|_{\partial\omega_h} = 0$ and $w^h = 0|_{\partial\omega_h}$ we have*

$$\frac{1}{2}\left(\left(u^h_t\right)^2, \frac{1}{Q^h} \right)_h + \frac{1}{2}\frac{d}{dt}\left(\left(w^h\right)^2, \frac{1}{Q^h} \right)_h = 0.$$

Proof. We start with the equation for w_{ij} (3.3), divide by Q^h_{ij}, multiply by ξ_{ij} vanishing on $\partial\omega_h$ and sum over ω.

$$\left(\frac{w^h}{Q^h},\xi\right)_h = \left(\left(\frac{u^h{}_{\overline{x}_1}}{Q^h}\right)_{x_1} + \left(\frac{u^h{}_{\overline{x}_2}}{Q^h}\right)_{x_2},\xi\right)_h .$$

The Green theorem (3.1) gives

$$\left(\frac{w^h}{Q^h},\xi\right)_h = -\left(\xi_{\overline{x}_1},\frac{u^h{}_{\overline{x}_1}}{Q^h}\right] - \left(\xi_{\overline{x}_2},\frac{u^h{}_{\overline{x}_2}}{Q^h}\right] \tag{3.7}$$

$$+\sum_{j=1}^{N_2-1}\left(\xi\frac{u^h{}_{\overline{x}_1}}{Q^h}\,|_{N_1 j} - \xi\frac{u^h{}_{\overline{x}_1}}{Q^h}\,|_{0j}\right)h_2, \tag{3.8}$$

$$+\sum_{i=1}^{N_1-1}\left(\xi\frac{u^h{}_{\overline{x}_2}}{Q^h}\,|_{iN_2} - \xi\frac{u^h{}_{\overline{x}_2}}{Q^h}\,|_{i0}\right)h_1, \tag{3.9}$$

and the terms (3.8) and (3.9) are equal to zero because of the choice of ξ_{ij}. Rewriting the equation (3.2) in the following form

$$\frac{u^h_t}{Q^h} = -\nabla_h\left(\frac{2}{Q^h}\mathbb{E}^h\overline{\nabla}_h w^h - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3}\overline{\nabla}_h u^h\right),$$

multiplying by test function φ vanishing at $\partial\omega_h$ and applying the Green theorem (3.1) we obtain

$$\left(\frac{u^h_t}{Q^h},\varphi\right)_h = \left(\frac{2}{Q^h}\mathbb{E}^h\overline{\nabla}_h w^h - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3}\overline{\nabla}_h u^h, \overline{\nabla}_h\varphi\right] \tag{3.10}$$

$$+\sum_{j=1}^{N_2-1}\left[\varphi\cdot\left(\mathbb{E}^h_{11}\cdot w^h{}_{\overline{x}_1} + \mathbb{E}^h_{12}\cdot w^h{}_{\overline{x}_2} - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3}u^h{}_{\overline{x}_1}\right)|_{N_1 j}\right. \tag{3.11}$$

$$\left.-\varphi\cdot\left(\mathbb{E}^h_{11}\cdot w^h{}_{\overline{x}_1} + \mathbb{E}^h_{12}\cdot w^h{}_{\overline{x}_2} - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3}u^h{}_{\overline{x}_1}\right)|_{0j}\right] \tag{3.12}$$

$$+\sum_{i=1}^{N_1-1}\left[\varphi\cdot\left(\mathbb{E}^h_{21}\cdot w^h{}_{\overline{x}_1} + \mathbb{E}^h_{22}\cdot w^h{}_{\overline{x}_2} - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3}u^h{}_{\overline{x}_2}\right)|_{iN_2}\right. \tag{3.13}$$

$$\left.-\varphi\cdot\left(\mathbb{E}^h_{21}\cdot w^h{}_{\overline{x}_1} + \mathbb{E}^h_{22}\cdot w^h{}_{\overline{x}_2} - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3}u^h{}_{\overline{x}_2}\right)|_{i0}\right]. \tag{3.14}$$

The terms (3.11), (3.12), (3.13) and (3.14) are zero because of φ_{ij} vanishing on $\partial\omega_h$. Differentiating (3.7) with respect to t we obtain

$$\frac{d}{dt}\left(\frac{w^h}{Q^h},\xi\right)_h + \frac{d}{dt}\left(\frac{u^h{}_{\overline{x}_1}}{Q^h},\xi_{\overline{x}_1}\right] + \frac{d}{dt}\left(\frac{u^h{}_{\overline{x}_2}}{Q^h},\xi_{\overline{x}_2}\right] = 0,$$

and using the following statements

$$\frac{d}{dt}\left(\frac{u^h{}_{\overline{x}_i}}{Q^h}\right) = \frac{\left(u^h{}_{\overline{x}_i}\right)_t Q - Q^h_t u^h{}_{\overline{x}_i}}{\left(Q^h\right)^2},\ i=1,2,$$

$$Q^h_t = \frac{\left(u^h{}_{\overline{x}_1}\right)_t u^h{}_{\overline{x}_1} + \left(u^h{}_{\overline{x}_2}\right)_t u^h{}_{\overline{x}_2}}{Q^h}$$

we get

$$\frac{d}{dt}\left(\frac{\overline{\nabla}_h u^h}{Q^h}\right)=\frac{\left(\left({u^h}_{\overline{x}_1}\right)_t,\left({u^h}_{\overline{x}_2}\right)_t\right)}{Q}-$$

$$-\frac{1}{\left(Q^h\right)^3}\cdot\left(\left({u^h}_{\overline{x}_1}\right)^2\left({u^h}_{\overline{x}_1}\right)_t+{u^h}_{\overline{x}_1}{u^h}_{\overline{x}_2}\left({u^h}_{\overline{x}_2}\right)_t,{u^h}_{\overline{x}_1}{u^h}_{\overline{x}_2}\left({u^h}_{\overline{x}_1}\right)_t+\left({u^h}_{\overline{x}_2}\right)^2\left({u^h}_{\overline{x}_2}\right)_t\right)=$$

$$=\frac{\left(\left({u^h}_{\overline{x}_1}\right)_t,\left({u^h}_{\overline{x}_2}\right)_t\right)}{Q^h}-\frac{1}{Q^h}\cdot\begin{pmatrix}\frac{\left({u^h}_{\overline{x}_1}\right)^2}{\left(Q^h\right)^2} & \frac{{u^h}_{\overline{x}_1}{u^h}_{\overline{x}_2}}{\left(Q^h\right)^2}\\ \frac{{u^h}_{\overline{x}_1}{u^h}_{\overline{x}_2}}{\left(Q^h\right)^2} & \frac{\left({u^h}_{\overline{x}_2}\right)^2}{\left(Q^h\right)^2}\end{pmatrix}\begin{pmatrix}\left({u^h}_{\overline{x}_1}\right)_t\\ \left({u^h}_{\overline{x}_2}\right)_t\end{pmatrix}=$$

$$=\frac{1}{Q^h}\left(\mathbb{I}-\mathbb{P}^h\right)\begin{pmatrix}\left({u^h}_{\overline{x}_1}\right)_t\\ \left({u^h}_{\overline{x}_2}\right)_t\end{pmatrix}=\mathbb{E}^h\begin{pmatrix}\left({u^h}_{\overline{x}_1}\right)_t\\ \left({u^h}_{\overline{x}_2}\right)_t\end{pmatrix},$$

which together with

$$\frac{d}{dt}\left(\frac{w^h}{Q^h}\right)=\frac{w^h_t}{Q^h}-\frac{Q^h_t\cdot w^h}{\left(Q^h\right)^2},$$

gives

$$\frac{d}{dt}\left(\frac{w^h}{Q^h},\xi\right)_h+\frac{d}{dt}\left(\frac{{u^h}_{\overline{x}_1}}{Q^h},\xi_{\overline{x}_1}\right]+\frac{d}{dt}\left(\frac{{u^h}_{\overline{x}_2}}{Q^h},\xi_{\overline{x}_2}\right]=$$

$$=\left(\frac{w^h_t}{Q^h},\xi\right)_h-\left(\frac{Q^h_t\cdot w^h}{\left(Q^h\right)^2},\xi\right)_h+\frac{1}{2}\left(\mathbb{E}^h\begin{pmatrix}\left({u^h}_{\overline{x}_1}\right)_t\\ \left({u^h}_{\overline{x}_2}\right)_t\end{pmatrix}\begin{pmatrix}\xi_{\overline{x}_1}\\ \xi_{\overline{x}_2}\end{pmatrix}\right]=0.$$

After substituting $\xi=w^h$ we obtain

$$\left(\frac{w^h_t}{Q^h},w^h\right)_h-\left(\frac{Q^h_t}{\left(Q^h\right)^2},\left(w^h\right)^2\right)_h+\frac{1}{2}\left(\mathbb{E}^h\overline{\nabla}_h u^h_t,\overline{\nabla}_h w^h\right]=0, \tag{3.15}$$

and a substitution $\varphi=u^h_t$ in (3.10) gives

$$\left(\left(u^h_t\right)^2,\frac{1}{\overline{Q}^h}\right)_h-\left(\mathbb{E}^h\overline{\nabla}_h w^h-\frac{\left(w^h\right)^2}{\left(Q^h\right)^3}\overline{\nabla}_h u^h,\overline{\nabla}_h u^h_t\right]=0. \tag{3.16}$$

Now we sum (3.15) with one half times (3.16) and we have

$$\left(\frac{w^h_t}{Q^h},w^h\right)_h-\left(\frac{Q^h_t}{\left(Q^h\right)^2},\left(w^h\right)^2\right)_h+\frac{1}{2}\left(\left(u^h_t\right)^2,\frac{1}{\overline{Q}^h}\right)_h+\frac{1}{2}\left(\frac{\left(w^h\right)^2}{\left(Q^h\right)^3},\overline{\nabla}_h u^h\cdot\overline{\nabla}_h u^h_t\right]=0.$$

We remind that $\overline{\nabla}_h u^h\cdot\overline{\nabla}_h u^h_t=Q^h\cdot Q^h_t$ which gives

$$\left(\frac{w^h_t}{Q^h},w^h\right)_h-\left(\frac{Q^h_t}{\left(Q^h\right)^2},\left(w^h\right)^2\right)_h+\frac{1}{2}\left(\left(u^h_t\right)^2,\frac{1}{\overline{Q}^h}\right)_h+\frac{1}{2}\left(\frac{\left(w^h\right)^2}{\left(Q^h\right)^2},Q^h_t\right]=0.$$

It is equivalent to

$$\frac{1}{2}\left(\left(u^h_t\right)^2,\frac{1}{\overline{Q}^h}\right)_h+\left(\frac{w^h_t}{\left(Q^h\right)^2},w^h\right)_h-\frac{1}{2}\left(\frac{Q^h_t}{\left(Q^h\right)^2},\left(w^h\right)^2\right)_h+S_h=0, \tag{3.17}$$

for

$$S_h = \frac{1}{2} \sum_{j=1}^{N_2-1} \left(\frac{w^h_{N_1 j}}{Q^h_{N_1 j}} \right)^2 \cdot \left(Q^h_t \right)_{N_1 j} h_1 h_2 + \frac{1}{2} \sum_{i=1}^{N_1-1} \left(\frac{w^h_{i N_2}}{Q^h_{i N_2}} \right)^2 \cdot \left(Q^h_t \right)_{i N_2} h_1 h_2.$$

Finaly from (3.17) we have

$$\frac{1}{2} \left(\left(u^h_t \right)^2, \frac{1}{Q^h} \right)_h + \frac{1}{2} \frac{d}{dt} \left(\left(w^h \right)^2, \frac{1}{Q^h} \right)_h + S_h = 0.$$

To complete the proof of we need to eliminate the term S_h. This can be done by applying the Dirichlet boundary conditions (2.13). □

Remark: The above given procedure can be used even for nonhomogenous time independent Dirichlet boundary conditions for u. Similar statetment as (3.2) for the Neumann boundary conditions remains an open problem.

4. Computational results. Here, we present several numerical experiments qualitative character. Quantitative results are summarized in [10]. First three examples show a decay towards planar surface. For all of them we considered homogeneous Dirichlet boundary conditions for u and w. Fig. 6.1 shows evolution of the initial condition $u_{ini}(x, y) = \sin(2\pi x) \cdot \sin(2\pi y)$ on domain $\Omega \equiv (0, 1)^2$ with 50×50 meshes and the space steps $h_1 = h_2 = 0.02$. The computation has been performed until the time $T = 0.01$.

In the Fig. 6.2 we show again a decay towards a planar surface. The initial condition is discontinuous: $u_{ini}(x, y) = \operatorname{sign}\left(x^2 + y^2 - 0.2^2\right)$. The domain Ω is $(-1, 1)^2$ and there are again 50×50 meshes and $h_1 = h_2 = 0.04$. We stopped the computation at the time $T = 1$.

The Fig. 6.3 shows a decay towards the planar surface with highly oscilating intial condition $u_{ini}(x, y) = \sin\left[2\pi \left(15 \tanh\left(\sqrt{x^2 + y^2} - 0.2\right)\right)\right]$. The domain Ω is $(-1, 1)^2$ and there are 50×50 meshes and $h_1 = h_2 = 0.04$. The final time for the computation was $T = 0.1$.

Next two examples show the restoration of a spherical surface. We start with a part of the sphere with radius $R = 3$ and center $C = (0, 0, -1.5)$ above the square domain $\Omega \equiv (-1, 1)^2$. We obtain a graph which can be described by a function u_S. It yields $w_S = Q(u_s) H(u_S)$. Then the following Dirichlet boundary conditions

$$u \mid_{\partial\omega_h} = u_S, \quad w \mid_{\partial\omega_h} = w_S,$$

are considered (they are more general than (2.13) and (2.14)). In case of Fig. 6.4 we perturb the original function u_S as follows

$$u_{ini} = u_S + \exp^{-5r} \cdot \sin(7.5\pi r),$$

for $r = \sqrt{x^2 + y^2}$. The initial condition for Fig. 6.5 was obtained by applying the heat equation on the initial function $v_{ini} \equiv 0$ with the Dirichlet boundary conditions $v \mid_{\partial\omega_h} = u_S$ and setting $u_{ini} = v \mid_{t=0.1}$. There were 50×50 meshes and $h_1 = h_2 = 0.04$ in both cases. In the first case (Fig. 6.4) we stopped the computation at the time $T = 0.05$ and in the second case (Fig. 6.5) at $T = 0.2$.

The example on Fig. 6.6 shows a computation with the homogeneous Neumann boundary conditions. The initial condition is $u_0 = \sin(2\pi x)$ on $\Omega = (0, 1)^2$ with 25×25 meshes and $h_1 = h_2 = 0.04$. The final time $T = 0.5$.

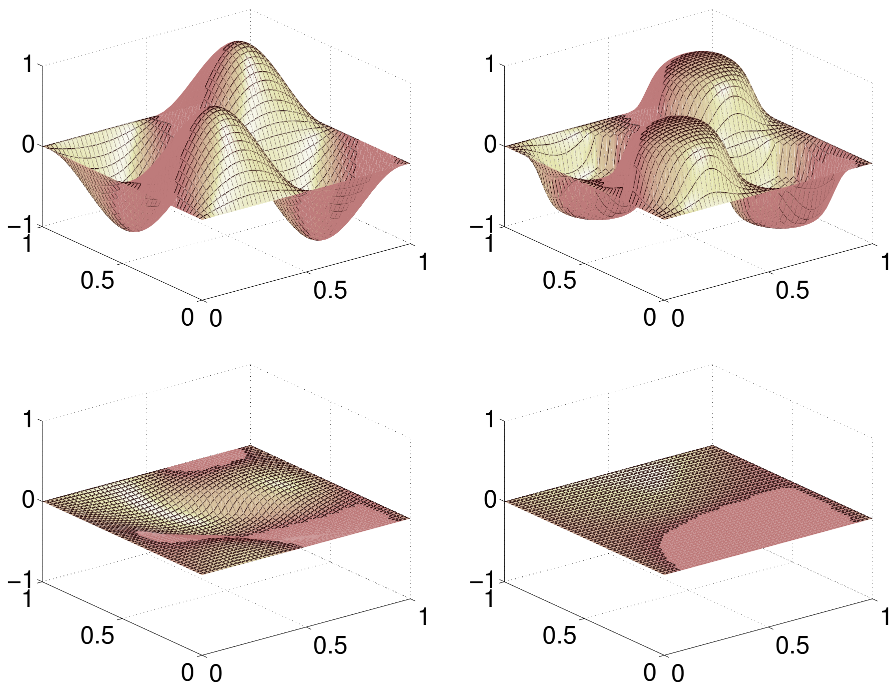

Fig. 6.1. *Convergence towards the planar surface at times* $t = 0$, $t = 10^{-4}$, $t = 17 \cdot 10^{-4}$ *and* $t = 0.01$.

5. Conclusion. In this article, we discussed a formulation of the Willmore flow for graphs and we presented a numerical scheme based on the method of lines. We have proved energy equality for the scheme and we have showed several computational experiments.

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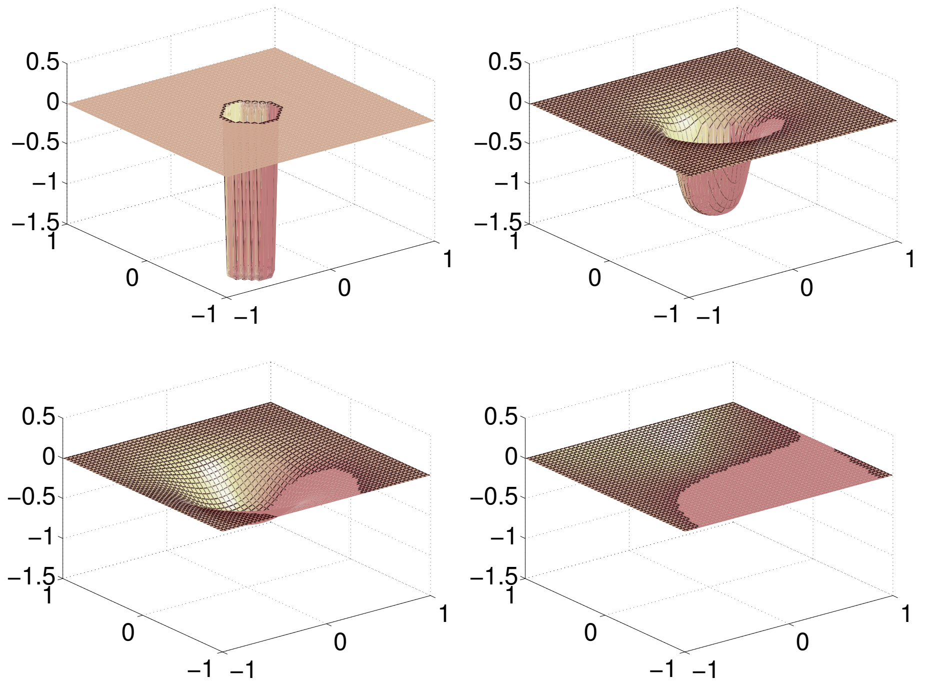

Fig. 6.2. *Convergence towards the planar surface at times $t = 0$, $t = 0.002$, $t = 0.005$ and $t = 1$.*

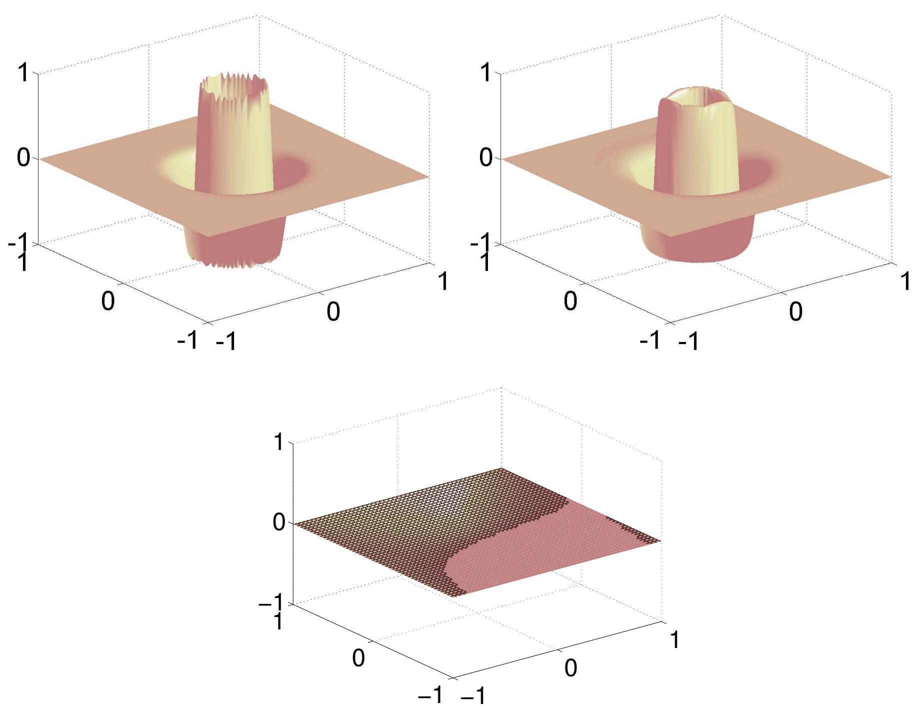

Fig. 6.3. *Convergence towards the planar surface at times $t = 0$, $t = 5 \cdot 10^{-6}$ and $t = 0.1$.*

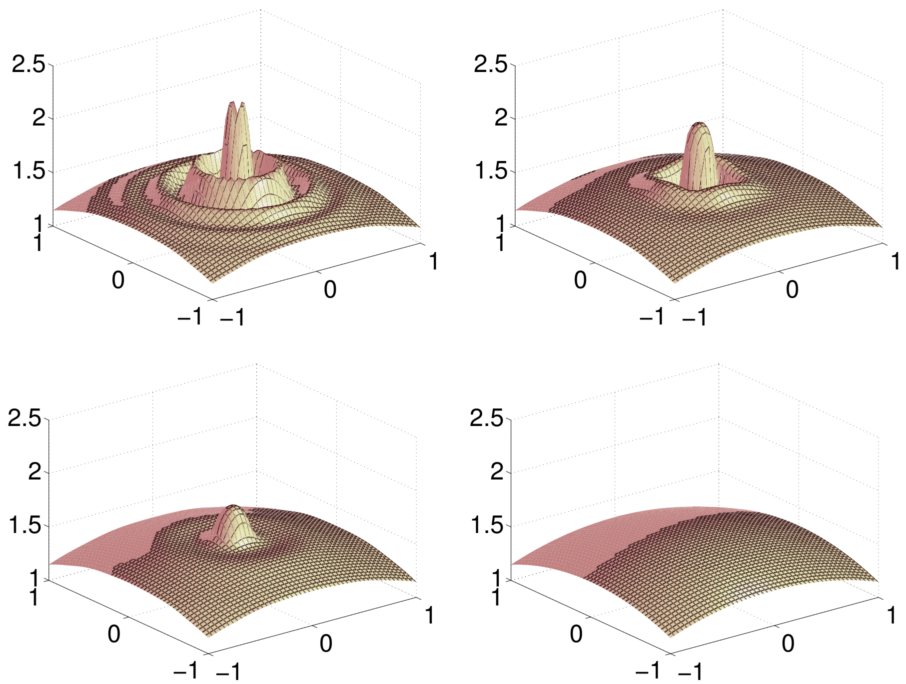

FIG. 6.4. *Spherical surface restoration at times* $t = 0$, $t = 2 \cdot 10^{-5}$, $t = 10^{-4}$ *and* $t = 0.05$.

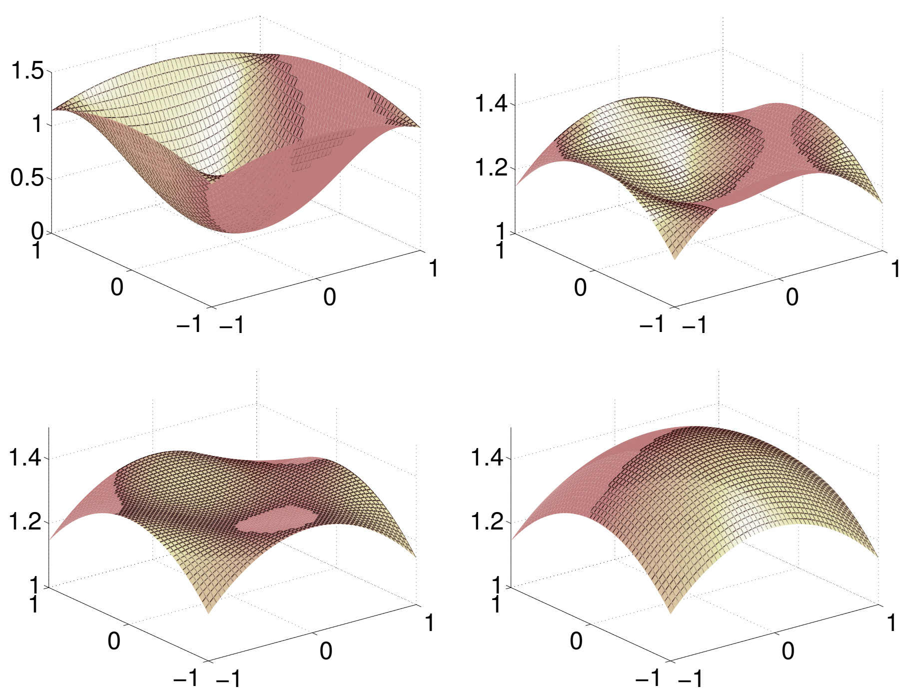

FIG. 6.5. *Spherical surface restoration at times* $t = 0$, $t = 0.05$, $t = 0.06$ *and* $t = 0.2$.

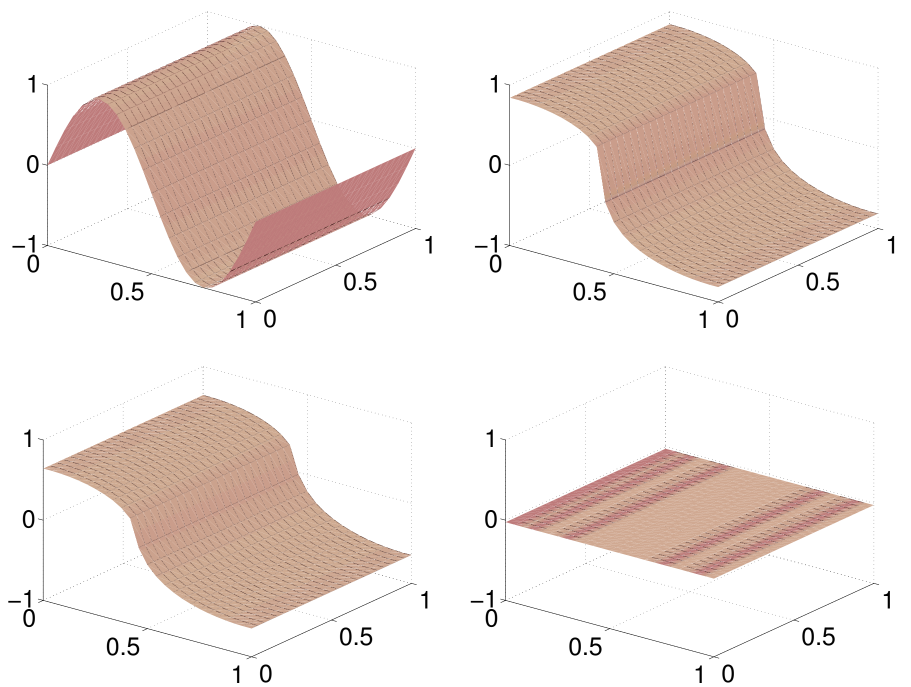

FIG. 6.6. *Test with the Neumann boundary conditions at times* $t = 0$, $t = 0.005$, $t = 0.175$ *and* $t = 0.5$.

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