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FINITE DIFFERENCE SCHEME FOR THE WILLMORE FLOW OF GRAPHS

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In this article we discuss numerical scheme for the approximation of the Willmore flow of graphs. The scheme is based on the finite difference method. We improve the scheme we presented in [8, 7] which is based on combination of the forward and the backward finite differences. The new scheme approximates the Willmore flow by the central differences and as a result it better preserves symmetry of the solution. Since it requires higher regularity of the solution, additional numerical viscosity is necessary in some cases. We also present theorem showing stability of the scheme together with the EOC and several results of the numerical experiments.

Keywords: Willmore flow, method of lines, curvature minimization, gradient flow, Laplace-Beltrami operator, Gauss curvature, central differences, numerical viscosity.
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1. INTRODUCTION

The Willmore flow is an evolutionary law for minimizing mean curvature of curves or surfaces. Consider a surface Γ_0 smooth enough so that at each point of this surface we can evaluate mean curvature $H = \kappa_1 + \kappa_2$ where κ_1 and κ_2 are the principal curvatures of the surface. Then we can define the Willmore functional $\mathcal{W}$ as

$$\mathcal{W}_{\Gamma_0} = \int_{\Gamma_0} H^2 dS. \tag{1}$$

This functional also expresses elastic energy of the surface. As follows from [1, 2, 5] the Willmore flow defined as

$$V = 2\triangle_\Gamma H + H^3 - 4HK \text{ on } \Gamma(t), \tag{2}$$

drives the surface towards the minimizer of (1). In (2) V is normal velocity, $\triangle_\Gamma$ is the Laplace-Beltrami operator and $K = \kappa_1 \cdot \kappa_2$ is the Gauss curvature.

Recently the Willmore flow has attracted interest of many mathematicians. Today several different formulations of the Willmore flow and different approaches to approximate the exact solution numerically are known. In [1] the authors study graph formulation of the Willmore flow and they apply the method of finite elements for discretization. Detailed numerical analysis can be found in their article as well. In [2] the level set formulation is derived and several numerical results obtained by the finite element method are presented. Numerical analysis of the level set formulation is difficult and no results have been published yet. Asymptotical convergence of the phase-field model for the Willmore flow has been proved in [3, 4]. For the approximation the authors chose the finite difference method. Finally in [5] the Lagrangian formulation of the elastic curves is studied. For the readers interested in the theory of the Willmore flow, we refer to [10, 9, 6].

2. PROBLEM FORMULATION

We assume that $\Gamma(t)$ is a graph of a function u of two variables:

$$\Gamma(t) = \left\{ [\mathbf{x}, u(t,\mathbf{x})] \mid \mathbf{x} \in \Omega \subset \mathbb{R}^2 \right\},$$

where $\Omega \equiv (0, L_1) \times (0, L_2)$ is an open rectangle, $\partial\Omega$ its boundary and ν its outer normal. Let us denote

$$Q = \sqrt{1 + |\nabla u|^2}, \quad \mathbb{E} = \frac{1}{Q}\left(\mathbb{I} - \frac{\nabla u}{Q} \otimes \frac{\nabla u}{Q}\right), \quad H = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right). \tag{3}$$

For the following definitions we refer to [8].

Definition 1 *The graph formulation for the Willmore flow is a system of two partial differential equations of the second order for u and w in the form*

$$\frac{\partial u}{\partial t} = -Q\nabla \cdot \left[2\mathbb{E}\nabla w - \frac{w^2}{Q^3}\nabla u\right] \text{ in } \Omega \times (0, T), \tag{4}$$

$$w = QH, \tag{5}$$

$$u(\cdot, 0) = u_{ini},$$

with the Dirichlet boundary conditions

$$u\mid_{\partial\Omega} = 0, \quad w\mid_{\partial\Omega} = 0, \tag{6}$$

or with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}\mid_{\partial\Omega} = 0, \quad \frac{\partial w}{\partial \nu}\mid_{\partial\Omega} = 0. \tag{7}$$

3. NUMERICAL SCHEME

For the numerical solution of (4)-(7), we will use method of lines with finite difference discretization in space.

We use the following notation. Let h_1, h_2 be space steps such that $h_1 = \frac{L_1}{N_1}$ and $h_2 = \frac{L_2}{N_2}$ for some $N_1, N_2 \in \mathbb{N}^+$. We define a uniform grid as

$$\begin{aligned} \omega_h &= \{(ih_1, jh_2) \mid i = 1 \cdots N_1 - 1, j = 1 \cdots N_2 - 1\}, \\ \overline{\omega}_h &= \{(ih_1, jh_2) \mid i = 0 \cdots N_1, j = 0 \cdots N_2\}, \\ \partial\omega_h &= \overline{\omega_h} \setminus \omega_h. \end{aligned}$$

For $u^h : \mathbb{R}^2 \to \mathbb{R}$ we define a projection on $\overline{\omega}_h$ as $u^h_{ij} = u(ih_1, jh_2)$. We introduce the differences as follows

$$
\begin{aligned}
u^h_{f.,ij} &= \frac{u_{i+1j} - u_{i,j}}{h_1} \quad , \quad u^h_{b.,ij} = \frac{u_{i,j} - u_{i-1j}}{h_1}, \\
u^h_{.f,ij} &= \frac{u_{ij+1} - u_{i,j}}{h_2} \quad , \quad u^h_{.b,ij} = \frac{u_{i,j} - u_{ij-1}}{h_2}, \\
u^h_{.c,ij} &= \frac{u^h_{.f,ij} + u^h_{.b,ij}}{2} \quad , \quad u^h_{c.,ij} = \frac{u^h_{f.,ij} + u^h_{b.,ij}}{2},
\end{aligned}
$$

and for the gradient approximation we will use the following notation (approximation of the divergence is done in the same manner)

$$
\nabla^h_f u^h_{ij} = \left(u^h_{f.,ij}, u^h_{.f,ij}\right) \quad , \quad \nabla^h_b u^h_{ij} = \left(u^h_{b.,ij}, u^h_{.b,ij}\right), \tag{8}
$$

$$
\nabla^h_c u^h_{ij} = \frac{1}{2} \left(\nabla^h_f u^h_{ij} + \nabla^h_b u^h_{ij}\right). \tag{9}
$$

For $f, g : \overline{\omega}_h \to \mathbb{R}$, $\boldsymbol{f}, \boldsymbol{g} : \overline{\omega}_h \to \mathbb{R}^2$ we define

$$
\begin{aligned}
(f, g)_h &= \sum_{i,j=1}^{N_1-1, N_2-1} h_1 h_2 f_{ij} g_{ij} \quad , \quad \|f\|^2_h = (f, f)_h, \\
\left(\boldsymbol{f}^1, \boldsymbol{g}^1\right)_{f.} &= \sum_{i,j=1}^{N_1, N_2-1} h_1 h_2 \boldsymbol{f}^1_{ij} \boldsymbol{g}^1_{ij} \quad , \quad \left(\boldsymbol{f}^2, \boldsymbol{g}^2\right)_{.f} = \sum_{i,j=1}^{N_1-1, N_2} h_1 h_2 \boldsymbol{f}^2_{ij} \boldsymbol{g}^2_{ij}, \\
\left(\boldsymbol{f}^1, \boldsymbol{g}^1\right)_{b.} &= \sum_{i=0,j=1}^{N_1-1, N_2-1} h_1 h_2 \boldsymbol{f}^1_{ij} \boldsymbol{g}^1_{ij} \quad , \quad \left(\boldsymbol{f}^2, \boldsymbol{g}^2\right)_{.b} = \sum_{i=1,j=0}^{N_1-1, N_2-1} h_1 h_2 \boldsymbol{f}^2_{ij} \boldsymbol{g}^2_{ij},
\end{aligned}
$$

$$
\begin{aligned}
\left(\boldsymbol{f}^1, \boldsymbol{g}^1\right)_{c.} &= \frac{1}{2} \left[\left(\boldsymbol{f}^1, \boldsymbol{g}^1\right)_{f.} + \left(\boldsymbol{f}^1, \boldsymbol{g}^1\right)_{b.}\right], \\
\left(\boldsymbol{f}^2, \boldsymbol{g}^2\right)_{.c} &= \frac{1}{2} \left[\left(\boldsymbol{f}^2, \boldsymbol{g}^2\right)_{.f} + \left(\boldsymbol{f}^2, \boldsymbol{g}^2\right)_{.b}\right], \\
\left(\boldsymbol{f}, \boldsymbol{g}\right)_c &= \left(\boldsymbol{f}^1, \boldsymbol{g}^1\right)_{c.} + \left(\boldsymbol{f}^2, \boldsymbol{g}^2\right)_{.c}.
\end{aligned}
$$

For the discretization of the Neumann boundary conditions we define the grid boundary normal difference $u_{\bar{n}}$:

$$
\begin{aligned}
u_{\bar{n},0j} &= u_{b.,1j} && \text{for } j = 0, \ldots, N_2, \\
u_{\bar{n},N_1j} &= u_{f.,N_1j} && \text{for } j = 0, \ldots, N_2, \\
u_{\bar{n},i0} &= u_{.b,i1} && \text{for } i = 0, \ldots, N_1, \\
u_{\bar{n},iN_2} &= u_{.f,iN_2} && \text{for } i = 0, \ldots, N_1.
\end{aligned}
$$

For the purpose of analysis, we will need the grid version of the Green formula:

Lemma 2 *Let $p, f, g : \bar{\omega}_h \rightarrow \mathbb{R}$. Then the following Green formulas are valid*

$$
\begin{aligned}
\left(\nabla^h_f \cdot \left(p \nabla^h_b f\right), g\right)_h &= -(p\nabla^h_b f, \nabla^h_b g)_f \\
&+ \sum_{j=1}^{N_2-1} h_2 (p f_{b.,N_1 j} g_{N_1 j} - p f_{b.,1j} g_{0j}) \\
&+ \sum_{i=1}^{N_1-1} h_1 (p f_{.b,iN_2} g_{iN_2} - p f_{.b,i1} g_{i0}).
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\left(\nabla^h_b \cdot \left(p \nabla^h_f f\right), g\right)_h &= -(p\nabla^h_f f, \nabla^h_f g)_b \\
&+ \sum_{j=1}^{N_2-1} h_2 (p f_{b.,N_1-1j} g_{N_1 j} - p f_{b.,0j} g_{0j}) \\
&+ \sum_{i=1}^{N_1-1} h_1 (p f_{.b,iN_2-1} g_{iN_2} - p f_{.b,i0} g_{i0}).
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\left(\nabla^h_c \cdot \left(p \nabla^h_c f\right), g\right)_h &= -(p\nabla^h_c f, \nabla^h_c g)_c \\
&+\frac{1}{2} \sum_{j=1}^{N_2-1} h_2 (p f_{b.,N_1 j} g_{N_1 j} - p f_{b.,1j} g_{0j} + p f_{b.,N_1-1j} g_{N_1 j} - p f_{b.,0j} g_{0j}) \\
&+\frac{1}{2} \sum_{i=1}^{N_1-1} h_1 (p f_{.b,iN_2} g_{iN_2} - p f_{.b,i1} g_{i0} + p f_{.b,iN_2-1} g_{iN_2} - p f_{.b,i0} g_{i0}).
\end{aligned} \tag{12}
$$

Proof. Let us denote $\mathcal{L}_h \equiv \{ih \mid 0 \leq i \leq N\}$ for $0 < h \in \mathbb{R}$ and $N \in \mathbb{N}^+$. Then for functions $u, v : \mathcal{L}_h \rightarrow \mathbb{R}$ we define the following scalar products

$$
\begin{aligned}
(u,v)_f = \sum_{i=1}^{N} h u_i v_i, \quad (u,v)_b &= \sum_{i=0}^{N-1} h u_i v_i, \quad (u,v)_h = \sum_{i=1}^{N-1} h u_i v_i, \\
(u,v)_c &= \frac{1}{2}\left[(u,v)_f + (u,v)_b\right].
\end{aligned}
$$

Now we have

$$
\begin{aligned}
(u_f, v)_h &= \sum_{i=1}^{N-1} \frac{u_{i+1}-u_i}{h} v_i h = \sum_{i=1}^{N-1} u_i v_{i+1} - \sum_{i=1}^{N-1} u_i v_i \\
&= \sum_{i=2}^{N} u_i v_{i-1} - \sum_{i=1}^{N-1} u_i v_i = \sum_{i=2}^{N-1} u_i \left(v_{i-1} - v_i\right) + u_N v_{N-1} - u_1 v_1 \\
&= \sum_{i=2}^{N-1} u_i \left(v_{i-1} - v_i\right) + u_N v_N + u_N \left(v_{N-1} - v_N\right) - u_1 v_0 + u_1 \left(v_0 - v_1\right) \\
&= u_N v_N - u_1 v_0 - (u, v_b)_f .
\end{aligned}
$$

In the same way we can show $(u_b, v)_h = u_{N-1}v_N - u_0v_0 - (u, v_f)_b$. For the central differences we get

$$\begin{aligned}(u_c, v)_h &= \left(\frac{1}{2}(u_f + u_b), v\right)_h = \frac{1}{2}(u_f, v)_h + \frac{1}{2}(u_b, v)_h \\ &= -\frac{1}{2}(u, v_b)_f + u_nv_n - u_1v_0 - \frac{1}{2}(u, v_f)_b + u_{N-1}v_n - u_0v_0 \\ &= -(u, v_c)_c + u_nv_n - u_1v_0 + u_{N-1}v_n - u_0v_0.\end{aligned}$$

Now we can proceed to the Green formula (10)

$$\begin{aligned}(\nabla_f^h \cdot (p\nabla_b^h f), g)_h &= \left((pf_{b.})_{f.}, g\right)_h + \left((pf_{.b})_{.f}, g\right)_h \\ &= \sum_{j=1}^{N_2-1} h_2 \left((p_{.j}f_{b.,.j})_{f.}, g_{.j}\right)_h + \sum_{i=1}^{N_1-1} h_1 \left((p_{i.}f_{.b,i.})_{.f}, g_{i.}\right)_h \\ &= \sum_{j=1}^{N_2-1} h_2 \left(-(p_{.j}f_{b.,.j}, g_{b.,.j})_f + p_{N_1j}f_{b.,N_1j}g_{N_1j} - p_{1j}f_{b.,1j}g_{0j}\right) \\ &\quad + \sum_{i=1}^{N_1-1} h_1 \left(-(p_{i.}f_{.b,i.}, g_{.b,i.})_f + p_{iN_2}f_{.b,iN_2}g_{iN_2} - p_{i1}f_{.b,i1}g_{i0}\right) \\ &= -(pf_{b.}, g_{b.})_{f.} - (pf_{.b}, g_{.b})_{.f} + \sum_{j=1}^{N_2-1} h_2 \cdot (p_{N_1j}f_{b.,N_1j}g_{N_1j} - p_{1j}f_{b.,1j}g_{0j}) \\ &\quad + \sum_{i=1}^{N_1-1} h_1 \cdot (p_{iN_2}f_{.b,iN_2}g_{iN_2} - p_{i1}f_{.b,i1}g_{i0}) \\ &= -(p\nabla_b^h f, \nabla_b^h g)_f + \sum_{j=1}^{N_2-1} h_2 \cdot (p_{N_1j}f_{b.,N_1j}g_{N_1j} - p_{1j}f_{b.,1j}g_{0j}) \\ &\quad + \sum_{i=1}^{N_1-1} h_1 \cdot (p_{iN_2}f_{.b,iN_2}g_{iN_2} - p_{i1}f_{.b,i1}g_{i0})\end{aligned}$$

Similarly one can prove (11). In the case of the central differences (12) we just use the fact that

$$\left(\nabla_c^h \cdot \left(p\nabla_c^h f\right), g\right)_h = \frac{1}{2}\left[\left(\nabla_f^h \cdot \left(p\nabla_b^h f\right), g\right)_h + \left(\nabla_b^h \cdot \left(p\nabla_f^h f\right), g\right)_h\right].$$

□

Corollary 3 *Let* $p, f, g : \bar{\omega}_h \to \mathbb{R}$ *and* $v\,|_{\partial\omega} = 0$. *Then*

$$(\nabla_f^h \cdot (p\nabla_b^h f), g)_h = -\left(p\nabla_b^h f, \nabla_b^h g\right)_f, \quad (13)$$

$$(\nabla_b^h \cdot (p\nabla_f^h f), g)_h = -\left(p\nabla_f^h f, \nabla_f^h g\right)_b, \quad (14)$$

$$(\nabla_c^h \cdot (p\nabla_c^h f), g)_h = -\left(p\nabla_c^h f, \nabla_c^h g\right)_c. \quad (15)$$

Denoting

$$
\begin{aligned}
Q^h_{ij} &= \sqrt{1+|\nabla^h_c u_{ij}|^2}, \quad H^h_{ij} = \nabla^h_c \cdot \left(\frac{\nabla^h_c u_{ij}}{Q^h_{ij}}\right), && (16)\\
\mathbb{E}^h_{ij} &= \frac{1}{Q^h_{ij}}\left(\mathbb{I} - \frac{\nabla^h_c u_{ij}}{Q^h_{ij}} \otimes \frac{\nabla^h_c u_{ij}}{Q^h_{ij}}\right), && (17)\\
R_{visc} &= C_{visc} Q^h_{ij}\left(h_1^2 \left(u^h_{b.ij}\right)_{f.,ij} + h_2^2 \left(u^h_{.b,ij}\right)_{.f,ij}\right), && (18)
\end{aligned}
$$

for $i = 1, \cdots, N_1 - 1$ and $j = 1, \cdots, N_2 - 1$, the scheme has the following form

$$
\begin{aligned}
\frac{du^h_{ij}}{dt} &= -Q^h_{ij}\nabla^h_c \cdot \left(2\mathbb{E}^h_{ij}\nabla^h_c w^h_{ij} - \frac{\left(w^h_{ij}\right)^2}{\left(Q^h_{ij}\right)^3}\nabla^h_c u^h_{ij}\right) + R_{visc}, && (19)\\
w^h_{ij} &= Q^h_{ij} H^h_{ij}, && (20)
\end{aligned}
$$

for $i = 1, \cdots, N_1 - 1$, $j = 1, \cdots, N_2 - 1$. In (16), (17) and (19) for $i = 0$ resp. $j = 0$ we approximate ∇u by $u^h_{f.,ij}$ resp. $u^h_{.f,ij}$ and for $i = N_1$ resp. $j = N_2$ by $u^h_{b.,ij}$ resp. $u^h_{.b,ij}$. The same holds for the approximation of ∇w in (19).

We set the initial condition

$$u^h(0)\,|_{\overline{\omega}_h} = u_{ini},$$

and we consider either the Dirichlet boundary conditions

$$u^h\,|_{\partial\omega_h} = 0, \quad w^h\,|_{\partial\omega_h} = 0. \tag{21}$$

or the Neumann boundary conditions

$$u^h\,|_{\partial\omega_h} = 0, \quad w^h_{\vec{n}}\,|_{\partial\omega_h} = 0. \tag{22}$$

Remark: The employ of the central differences gives us a scheme with symmetric stencil. It is important advantage in comparison with the scheme using only the forward and backward differences. The disadvantage of the scheme (19)-(20) is that it tends to oscillate when approximating solution with lower regularity. The remedy of this problem is just in the term (18) which keeps the numerical approximation smooth enough. At this point we must note that we need to regularize a term of the fourth order. It is of much larger magnitude then the regularizing term (18) itself. This is why C_{visc} must be much larger in some situations then for example in the case of the Navier-Stokes equations.

The following theorem shows the energy equality of the scheme (for simplicity we assume $h_1 = h_2 = h$).

Theorem 4 *For $u^h\,|_{\partial\omega_h} = 0$ and $w^h = 0\,|_{\partial\omega_h}$ we have*

$$\left(\left(u^h_t\right)^2, \frac{1}{Q^h}\right)_h + \frac{d}{dt}\left[\left(\left(H^h\right)^2, Q^h\right)_h - C_{visc}\frac{h^2}{2}\left(\nabla^h_b u^h, \nabla^h_b u^h\right)_h\right] = 0.$$

Proof. We start with the equation for w_{ij}^h (20), divide by Q_{ij}^h, multiply by ξ_{ij} vanishing on $\partial\omega_h$ and sum over ω

$$\left(\frac{w^h}{Q^h}, \xi\right)_h = \left(\nabla_c^h \cdot \left(\frac{\nabla_c^h u^h}{Q^h}\right), \xi\right)_h .$$

The Green theorem (15) gives

$$\left(\frac{w^h}{Q^h}, \xi\right)_h = -\left(\frac{\nabla_c^h u^h}{Q^h}, \nabla_c^h \xi\right)_c . \tag{23}$$

Now consider the right hand side of (19), divide by Q^h, multiply by the test function φ vanishing at $\partial\omega_h$ and applying the Green theorem (15) we obtain

$$\begin{aligned}\left(-\nabla_c^h \cdot \left(2\mathbb{E}^h \nabla_c^h w^h - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3} \nabla_c^h u^h\right), \varphi\right)_h &= \\ \left(2\mathbb{E}^h \nabla_c^h w^h - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3} \nabla_c^h u^h, \nabla_c^h \varphi\right)_c &. \end{aligned} \tag{24}$$

Since

$$\left(Q_{ij}^h\right)_t = \frac{\left(\nabla_c^h u_{ij}^h\right)_t \cdot \nabla_c^h u_{ij}^h}{Q_{ij}^h} \tag{25}$$

we get

$$\begin{aligned}
\tfrac{d}{dt} \left(\frac{\nabla_c^h u^h}{Q^h}\right) &= \frac{\left(\nabla_c^h u^h\right)_t}{Q^h} - \\
&- \frac{1}{\left(Q_{ij}^h\right)^3} \cdot \left(\left(u_{c.}^h\right)^2 \left(u_{c.}^h\right)_t + u_{c.}^h u_{.c}^h \left(u_{.c}^h\right)_t, u_{c.}^h u_{.c}^h \left(u_{c.}^h\right)_t + \left(u_{.c}^h\right)^2 \left(u_{.c}^h\right)_t\right) = \\
&= \frac{\left(\left(u_{c.}^h\right)_t, \left(u_{.c}^h\right)_t\right)}{Q^h} - \frac{1}{Q^h} \cdot \begin{pmatrix} \frac{\left(u_{c.}^h\right)^2}{\left(Q^h\right)^2} & \frac{u_{c.}^h u_{.c}^h}{\left(Q^h\right)^2} \\ \frac{u_{c.}^h u_{.c}^h}{\left(Q^h\right)^2} & \frac{\left(u_{.c}^h\right)^2}{\left(Q^h\right)^2} \end{pmatrix} \begin{pmatrix} \left(u_{c.}^h\right)_t \\ \left(u_{.c}^h\right)_t \end{pmatrix} = \\
&= \frac{1}{Q^h} \left(\mathbb{I} - \mathbb{P}^h\right) \begin{pmatrix} \left(u_{c.}^h\right)_t \\ \left(u_{.c}^h\right)_t \end{pmatrix} = \mathbb{E}^h \nabla_c^h u_t^h .
\end{aligned}$$

Differentiating (23) with respect to t we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{w^h}{Q^h}, \xi\right)_h &+ \frac{d}{dt}\left(\frac{\nabla_c^h u^h}{Q^h}, \nabla_c^h \xi\right)_c \\
&= \left(\frac{w_t^h}{Q^h}, \xi\right)_h - \left(\frac{Q_t^h \cdot w^h}{\left(Q^h\right)^2}, \xi\right)_h + \left(\mathbb{E}^h \nabla_c^h u_t^h, \nabla_c^h \xi\right)_c = 0.
\end{aligned}$$

After substituting $\xi = w^h$ we obtain

$$\left(\frac{w_t^h}{Q^h}, w^h\right)_h - \left(\frac{Q_t^h}{\left(Q^h\right)^2}, \left(w^h\right)^2\right)_h + \left(\mathbb{E}^h \nabla_c^h u_t^h, \nabla_c^h w^h\right)_c = 0, \tag{26}$$

substitution $\varphi = u_t^h$ in (24) gives

$$\left(\left(u_t^h\right)^2, \frac{1}{Q^h}\right)_h - \left(2\mathbb{E}^h \nabla_c^h w^h - \frac{\left(w^h\right)^2}{\left(Q^h\right)^3} \nabla_c^h u^h, \nabla_c^h u_t^h\right)_c = 0. \tag{27}$$

Substituting (26) to (27) (term $\mathbb{E}^h \nabla_c^h w^h$) we have

$$\left(\left(u_t^h\right)^2, \frac{1}{Q^h}\right)_h + 2\left(\frac{w_t^h}{Q^h}, w^h\right)_h - 2\left(\frac{Q_t^h}{\left(Q^h\right)^2}, \left(w^h\right)^2\right)_h + \left(\frac{\left(w^h\right)^2}{\left(Q^h\right)^3}, \nabla_c^h u^h \cdot \nabla_c^h u_t^h\right)_c = 0. \tag{28}$$

Using (25) gives

$$\left(\left(u_t^h\right)^2, \frac{1}{Q^h}\right)_h + 2\left(\frac{w_t^h}{Q^h}, w^h\right)_h - 2\left(\frac{Q_t^h}{\left(Q^h\right)^2}, \left(w^h\right)^2\right)_h + \left(\frac{\left(w^h\right)^2}{\left(Q^h\right)^2}, Q_t^h\right)_c = 0 \tag{29}$$

Since w^h is vanishing on $\partial\omega_h$ we have

$$\left(\frac{Q_t^h}{\left(Q^h\right)^2}, \left(w^h\right)^2\right)_h = \left(\frac{\left(w^h\right)^2}{\left(Q^h\right)^2}, Q_t^h\right)_c,$$

and (29) is equivalent to

$$\left(\left(u_t^h\right)^2, \frac{1}{Q^h}\right)_h + 2\left(\frac{w_t^h}{\left(Q^h\right)^2}, w^h\right)_h - \left(\frac{Q_t^h}{\left(Q^h\right)^2}, \left(w^h\right)^2\right)_h = 0. \tag{30}$$

Rewriting the second and the third term in (30) using

$$2\left(\frac{w_t^h}{\left(Q^h\right)^2}, w^h\right)_h - \left(\frac{Q_t^h}{\left(Q^h\right)^2}, \left(w^h\right)^2\right)_h = \frac{d}{dt}\left(\left(w^h\right)^2, \frac{1}{Q^h}\right)_h = \frac{d}{dt}\left(\left(H^h\right)^2, Q^h\right)_h.$$

we end up with

$$\left(\left(u_t^h\right)^2, \frac{1}{Q^h}\right)_h + \frac{d}{dt}\left(\left(H^h\right)^2, Q^h\right)_h = 0.$$

For the viscose term R_{visc} we have $R_{visc} = C_{visc} h^2 \nabla_f^h \nabla_b^h u^h$. Multiplying by φ vanishing on $\partial\omega^h$ we get

$$\left(C_{visc} h^2 \nabla_f^h \nabla_b^h u^h, \varphi\right)_h = -C_{visc} h^2 \left(\nabla_b^h u^h, \nabla_b^h \varphi\right)_f = -C_{visc} h^2 \left(\nabla_b^h u^h, \nabla_b^h \varphi\right)_h.$$

The last equality holds since $\varphi\mid_{\partial\omega^h} = 0$. Setting $\varphi = u_t^h$ we obtain

$$-C_{visc} h^2 \left(\nabla_b^h u^h, \nabla_b^h u_t^h\right)_h = -C_{visc} \frac{h^2}{2} \frac{d}{dt} \left(\nabla_b^h u^h, \nabla_b^h u^h\right)_h.$$

□

Remark: Similar statement as (4) for the Neumann boundary conditions remains an open problem.

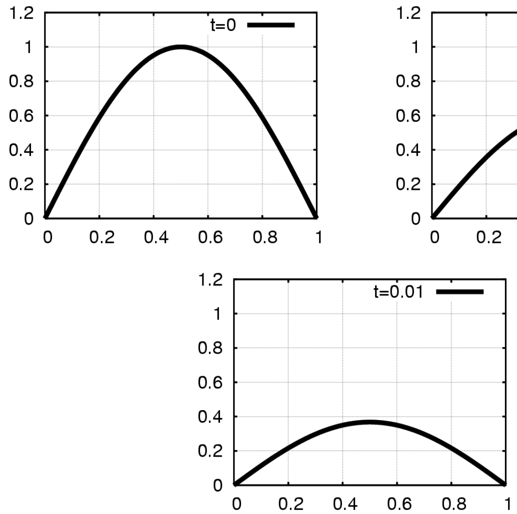

Figure 1: Decay towards a planar surface at times $t = 0$, $t = 0.005$ and $t = 0.01$.

4. EXPERIMENTAL ORDER OF CONVERGENCE

In this section we study the experimental order of convergence of the presented scheme. In the case of graphs there is not known any analytical solution for the Willmore flow. Therefore we modify the equation (4) in such way that it has an analytical solution. Suppose we want $u^\star(x,t) = \sin(\pi x) \cdot e^{-100t}$ to be solution of modified equation defined on $\Omega \equiv (0,1) \times (0,1)$ - see Fig. 1. It is easy to see that the modified equation takes the following form:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -Q\nabla \cdot \left[\frac{2}{Q} (\mathbb{I} - \mathbb{P}) \nabla w - \frac{w^2}{Q^3} \nabla u \right] \\ &\quad - \frac{\partial u^\star}{\partial t} + Q^\star \nabla \cdot \left[\frac{2}{Q^\star} (\mathbb{I} - \mathbb{P}^\star) \nabla w^\star - \frac{(w^\star)^2}{(Q^\star)^3} \nabla u^\star \right], \end{aligned}$$

for

$$Q^\star = \sqrt{1 + |\nabla u^\star|^2}, \quad \mathbb{P}^\star = \frac{\nabla u^\star}{Q^\star} \otimes \frac{\nabla u^\star}{Q^\star}, w^\star = Q^\star \nabla \frac{\nabla u^\star}{Q^\star}.$$

We set time dependent Dirichlet boundary conditions $u^h \mid_{x=0} = u_h \mid_{x=1} = 0$, $w^h \mid_{x=0} = w^* \mid_{x=0}$ and $w^h \mid_{x=1} = w^* \mid_{x=1}$ combined with the Neumann boundary conditions $\frac{\partial u^h}{\partial \nu} \mid_{y=0} = \frac{\partial u^h}{\partial \nu} \mid_{y=1} = \frac{\partial w^h}{\partial \nu} \mid_{y=0} = \frac{\partial w^h}{\partial \nu} \mid_{y=1} = 0$. We evaluate the following errors of

Meshes	h	EOC $E^{h_i}_{L_1(\Omega)}$	EOC $E^{h_i}_{L_2(\Omega)}$	EOC $E^{h_i}_{L_\infty(\Omega)}$
20	0.05	2.58969820958	2.4653462049	2.17520282298
30	0.03333	2.97637543595	3.61729905377	3.98116947809
40	0.025	2.29855758399	2.66369360864	3.54740507303
50	0.02	1.95805707937	2.11198673181	2.51711995978
60	0.01666	2.01897594977	2.04163094651	2.10533271594
70	0.01428	1.99159354982	2.02084609088	2.03726029417
80	0.0125	1.97100585415	2.00529812469	2.04216507258
90	0.01111	1.95921581197	1.99160335394	2.04564154453
100	0.01	1.95565974986	1.97895099513	1.9874281869

Figure 2: EOC for the scheme (19)-(20) evaluated by (31) - (33).

the evolution until time T with discretization parameter h_i:

$$E^{h_i}_{L_1(\Omega)} = \int_0^T \int_\Omega \left| u^h(x,t) - u^*(x,t) \right| dxdt, \tag{31}$$

$$E^{h_i}_{L_2(\Omega)} = \left(\int_0^T \int_\Omega \left(u^h(x,t) - u^*(x,t) \right)^2 dxdt \right)^{\frac{1}{2}}, \tag{32}$$

$$E^{h_i}_{L_\infty(\Omega)} = \max_{t \in \langle 0,T \rangle, x \in \Omega} \left| u^h(x,t) - u^*(x,t) \right|. \tag{33}$$

Related EOC for the scheme (19)-(20) with $C_{visc} = 10^3$ is in Fig. 2.

5. COMPUTATIONAL RESULTS

In this section we show several evolutions obtained by (19-20). On the Fig. 3 the initial condition is

$$u_0(x,y) = 0.5 \sin\left(\pi \tanh\left(5.0\left(x^2+y^2\right) - 0.25\right)\right)$$

on the domain $\Omega \equiv \langle -1,1 \rangle \times \langle -1,1 \rangle$. We set the Dirichlet boundary conditions $u \mid_{\partial\Omega} = w \mid_{\partial\Omega} = 0$. The steady state is the planar surface. On the Fig. 4 the initial condition is $u_0(x,y) = 0.5 \sin(5\pi\sqrt{x^2+y^2})$. The computational domain is the same as in the previous example. However in this experiment we set the Neumann boundary conditions $\frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = \frac{\partial w}{\partial \nu} \mid_{\partial\Omega} = 0$. On the Fig. 5, 6 the initial condition is $u_0(x,y) = \sin(2\pi x)$ on $\Omega \equiv \langle 0,1 \rangle \times \langle 0,1 \rangle$. For the Fig. 5 we set $\frac{\partial u}{\partial \nu} \mid_{\partial\Omega} = \frac{\partial w}{\partial \nu} \mid_{\partial\Omega} = 0$ and for the Fig. 6 we set $\frac{\partial u}{\partial \nu} \mid_{y=0,y=1} = 0$, $\frac{\partial u}{\partial \nu} \mid_{x=1} = -1$, $\frac{\partial u}{\partial \nu} \mid_{x=1} = 1$ and $\frac{\partial w}{\partial \nu} \mid_{\partial\Omega} = 0$.

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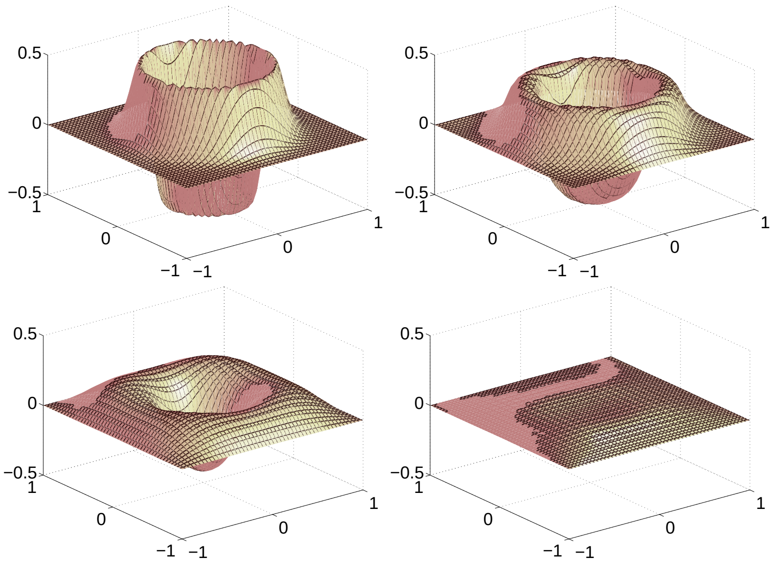

Figure 3: Convergence towards the planar surface with the Dirichlet boundary conditions and $C_{visc} = 0$ at times $t = 0$, $t = 2.5 \cdot 10^{-4}$, $t = 0.001875$ and $t = 0.005$.

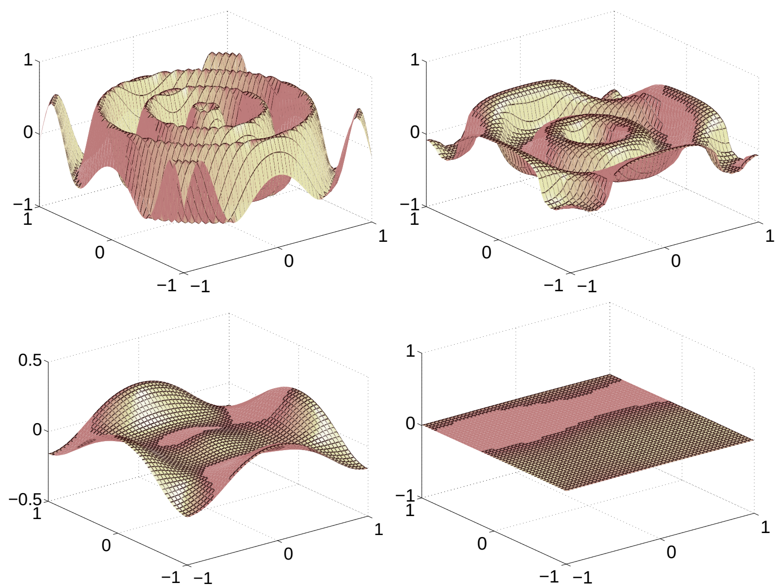

Figure 4: Convergence towards a planar surface with the Neumann boundary conditions and $C_{visc} = 0$ at times $t = 0$, $t = 5.0 \cdot 10^{-4}$, $t = 0.005$ and $t = 0.01$.

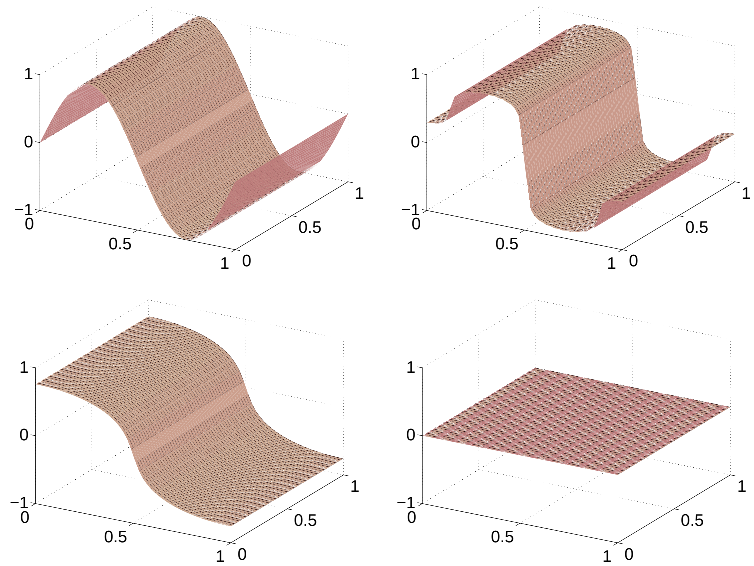

Figure 5: Test with the Neumann boundary conditions and $C_{visc} = 1000$ at times $t = 0$, $t = 0.001$, $t = 0.005$ and $t = 0.15$.

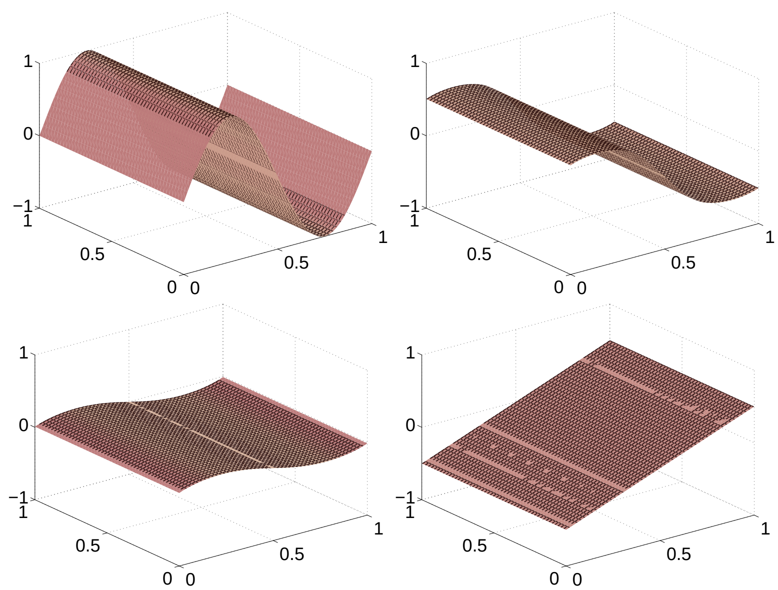

Figure 6: Test with the Neumann boundary conditions and $C_{visc} = 0$ at times $t = 0$, $t = 5.0 \cdot 10^{-4}$, $t = 0.0025$ and $t = 0.025$.

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