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CONLEY INDEX AND NUMERICAL VERIFICATION

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Abstract. The purpose of my work is to develop a rigorous numerical technique to prove the existence of stationary solutions and to detect connecting orbits among them in dissipative PDEs. The Conley index, topological quantities defined on invariant sets in dynamical systems, is applied to these problems. We consider the cubic Swift-Hohenberg equation as an example and study how to rigorously verify the existence of stationary solutions and connecting orbits among them. An effective algorithm by using FFT for this verification technique is also shown. This algorithm is applied to study the snaky bifurcation structure appearing in the quintic Swift-Hohenberg equation.

Key words. rigorous numerics, Conley index, FFT

AMS subject classifications. 35B45, 35B60, 37L25, 37B30

1. Introduction. One of the ultimate goals in dynamical systems is to study the structure of invariant sets in phase spaces. For instance, we often try to find equilibria, periodic orbits, homoclinic orbits, and so on in order to understand the underlying phenomena for a given dynamical system. In many cases, numerical approaches are regarded as one of the powerful tools for this purpose and many works which make use of computer simulations are performed to investigate complicated dynamical systems.

On the other hand, in recent years, a new concept, called numerical verification, is gradually getting popular. The numerical verifications mean that by using computers we obtain mathematically rigorous results. One of the key techniques for numerical verifications is the interval arithmetic. For example, Nakao's method [7] shows that sufficient conditions for the existence of solutions in elliptic PDEs can be checked by using computers with interval arithmetic.

Our main goal in this research is to develop a new rigorous numerical method to prove the existence of stationary solutions and to detect the connecting orbits among them in dissipative PDEs. We insist that although there are some kinds of numerical verification methods for the existence of stationary solutions, our method enables us to capture not only local stationary solutions but also the global dynamics. We use the Conley index, topological quantity defined on an invariant set, for the numerical verification and this topological approach plays a crucial role to detect connecting orbits.

This paper begins in Section 2 with an explanation of topological verification method to verify the existence of stationary solutions after a brief summary of the Conley index theory. Then in Section 3, we move on to the discussion about a rigorous numerical scheme to detect connecting orbits in gradient systems. Finally we study an efficient technique to reduce computational costs for the verification in Section 4.

2. Topological verification method. In this section, we begin with the brief introduction of the Conley index theory, which plays a central role throughout the paper. The general references are [1][6][9].

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Let X be a locally compact topological space and $\varphi : \mathbb{R} \times X \to X$ be a flow on X. A compact set $N \subset X$ is defined as an isolating neighborhood if the maximal invariant set of N is contained in the interior of N, i.e.

$$\mathrm{Inv}(\varphi, N) := \{x \in N \mid \varphi(\mathbb{R}, x) \subset N\} \subset \mathrm{Int}\ N.$$

This maximal invariant set $\mathrm{Inv}(\varphi, N)$ is called the isolated invariant set. Moreover, if the boundary of the isolating neighborhood is composed by the union of

$$L^+ := \{x \in \partial N \mid \exists t > 0 \ s.t.\ \varphi((0,t), x) \cap N = \emptyset\},$$
$$L^- := \{x \in \partial N \mid \exists t > 0 \ s.t.\ \varphi((-t,0), x) \cap N = \emptyset\}$$

then N is defined as an isolating block and L^+, L^- are called the exit set and the entrance set, respectively.

Definition 1. The Conley index of the isolated invariant set $\mathrm{Inv}(\varphi, N)$ is defined by

$$CH_* \left(\mathrm{Inv}(\varphi, N)\right) := H_* \left(N, L^+\right),$$

where $H_*(N, L^+)$ expresses the relative homology of (N, L^+).

We remark here that the above definition is well-defined [1][9]. That is to say, for any given isolated invariant set, there exist an isolating block and its exit set. In addition, if (N_i, L_i), $i = 1, 2$ are pairs of isolating blocks and these exit sets for the same isolated invariant set, then

$$CH_*(N_1, L_1^+) \cong CH_*(N_2, L_2^+).$$

In the paper [6], Conley indices for several isolated invariant sets are calculated.

Let us here discuss the relationship between stationary solutions of evolution equations and Conley indices. Our main purpose in this section is to obtain the rigorous numerical technique to prove the existence of stationary solutions by the information of Conley indices. We treat the Swift-Hohenberg equation with the periodic boundary condition:

$$u_t = E(u) := \left\{\nu - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right\} u - u^3, \tag{2.1}$$
$$u(x,t) = u(x + L_0, t), \quad u \in L^2(0, L_0)$$

as an example. Here we also assume $u(x,t) = u(-x,t)$. First of all, let us introduce the Fourier cosine expansion

$$u(x,t) = \sum_{j=0}^{\infty} a_j(t) \cos(jk_0 x),$$

where $k_0 = 2\pi/L_0$. Then the Swift-Hohenberg equation is expressed by

$$\dot{a}_j = f_j(a) = \zeta_j a_j - f_j^{(3)}(a), \quad j = 0, 1, \cdots, \tag{2.2}$$

where

$$\zeta_j = \nu - \left(1 - j^2 k_0^2\right)^2, \qquad f_j^{(3)}(a) = \sum_{\substack{m_1+m_2+m_3=j \\ m_i \in \mathbb{Z}}} a_{m_1} a_{m_2} a_{m_3}.$$

Hence, stationary solutions $E(u) = 0$ of (2.1) may be regarded as equilibria $f_j(a) = 0,\ j \geq 0$ of (2.2).

In this setting, we decompose the variable $a = \{a_0, a_1, \cdots\}$ and the vector field $f(a)$ into two parts for some positive integer m such as $a = (a_F, a_I),\ f(a) = (f_F(a), f_I(a))$ with

$$a_F = (a_0, a_1, \cdots, a_m), \quad a_I = (a_{m+1}, a_{m+2}, \cdots),$$
$$f_F(a) = (f_0(a), f_1(a), \cdots, f_m(a)), \quad f_I(a) = (f_{m+1}(a), f_{m+2}(a), \cdots).$$

In the rest of the paper, the subscripts F and I represent the finite part and the infinite part, respectively. Following this decomposition, we prepare the approximate solution $\bar{a} = (\bar{a}_F, 0)$ for the verification. Here $\bar{a}_F$ is an approximate solution, i.e. $g_F(\bar{a}_F) := f_F(\bar{a}_F, a_I = 0) \approx 0$. The approximate solutions in this paper are computed by the Galerkin method. Note that the finite part of the vector field can be expressed by $f_F(a_F, a_I) = g_F(a_F) + r(a_F, a_I)$, where $r(a_F, a_I)$ is the error term. Let us remark that the dimension m should be large enough to include the essential dynamics of the original differential equation around the equilibria.

Let us here introduce a new variable $\{b_j\}$ satisfying

$$(Pb_F + \bar{a}_F, b_I) = (a_F, a_I),$$

where the eigenvectors $p_j, j = 0, 1, \cdots, m$, of the Jacobi matrix $Dg_F(\bar{a}_F)$ are taken to be column vectors for the matrix P. By this transformation, we can deal with the local dynamics around the approximate solution in the neighborhood of the origin. This transformation is denoted by $T :\ (b_F, b_I) \mapsto (a_F, a_I)$. After the Taylor expansion of $g_F(a_F)$ at $\bar{a}_F$ with the new variable $\{b_j\}$, the original dynamical system can be represented by

$$\dot{b} = h(b), \tag{2.3}$$

where the finite part and infinite part of the vector field $\{h_j(b)\}$ are, respectively,

$$h_j(b) = \begin{cases} \lambda_j b_j + R_j(b), & j = 0, 1, \cdots, m, \\ f_j(Pb_F + \bar{a}_F, b_I), & j > m. \end{cases}$$

Here λ_j is the eigenvalue of the eigenvector p_j and $R_F(b)$ is given by

$$R_F(b) = P^{-1}\left(g_F(\bar{a}_F) + \frac{1}{2}D^2 g_F(\bar{a}_F)(Pb_F)^2 + \frac{1}{3!}D^3 g_F(\bar{a}_F)(Pb_F)^3 + r(b_F, b_I)\right).$$

For the simplicity of the explanation, we assume $\lambda_j (\neq 0) \in \mathbb{R}$ for all $j = 0, 1, \cdots, m$.

Definition 2. A compact set $W = \prod_{j\geq 0}\left[b_j^-, b_j^+\right] \ni 0$ is defined as a lifting set if the following conditions are satisfied.

1. The operator E is continuous on $X_W := \left\{u = \sum_{j\geq 0} a_j \cos(jk_0 x) \in X \mid\ a \in T \cdot W\right\}$
2. $W_F = \prod_{j=0}^{m}[b_j^-, b_j^+]$ is an isolating block for the flow $\varphi^{(b_I)}$ generated by $h_F(b_F, b_I),\ \forall b_I \in W_I = \prod_{j>m}[b_j^-, b_j^+]$
3. The boundary $W_F \times \partial W_I$ is an entrance set

We are now ready to introduce an important theorem.

Theorem 3. ([11]) Let W be a lifting set for the dynamical system (2.3). If the Conley index of W_F takes

$$CH_j\left(\mathrm{Inv}\left(W_F, \varphi^{(b_I)}\right)\right) \cong \begin{cases} \mathbb{Z}_2, & j = k, \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

for some $k \in \{0, 1, \cdots, m\}$, then there exists an equilibrium point of (2.2) in $T \cdot W$.

Therefore, from the viewpoint of the rigorous numerics, we need to construct a lifting set which satisfies the sufficient condition of Theorem 3. However, note that an infinite number of calculations, which are impossible for computers, are needed to check the condition 3 of the lifting set. Therefore, to reduce the infinite dimensional problem into the finite dimensional problem, we restrict lifting sets within appropriate forms. In this article, we assume the power decay property on the infinite part, i.e.

$$W = W_F \times W_I,$$
$$W_F := \prod_{j=0}^{m} \left[b_j^-, b_j^+\right], \qquad W_I := \prod_{j>m} \left[-\frac{c}{j^s}, \frac{c}{j^s}\right], \tag{2.5}$$

where c and s are positive constants. Note that the lifting set of this form can be expressed by the finite data (b_j^-, b_j^+, c, s) for $j = 0, 1, \cdots, m$.

It is known that, from the argument in [2][11], this setting of the lifting set enables us to obtain the estimates for the vector field $\{h_j(b)\}$ in W by using the interval arithmetic. Therefore we can rigorously check the sufficient condition in Theorem 3. The details of the algorithm for the verification and some numerical results can be found in [2].

3. Detection of connecting orbits. This section is devoted to presenting a method to detect connecting orbits among verified stationary solutions in gradient systems. This is one of the advantages to use topological method to verify the existence of stationary solutions. Through the algebraic argument with the information of Conley indices, we can investigate the global dynamics. To see the details, let me first briefly review several notions developed in the Conley index theory [1][3][9].

On the same setting at the beginning of Section 2, let us again suppose the flow $\varphi : \mathbb{R} \times X \to X$. For each $x \in X$, $\alpha(x)$ and $\omega(x)$ denote the α-limit set and the ω-limit set, respectively. We here consider the following decomposition of an isolated invariant set S.

Definition 4. A Morse decomposition of S is a finite collection

$$\mathcal{M}(S, <) := \{M(p) \mid p \in \mathcal{P}\}$$

of mutually disjoint invariant subsets $M(p)$ of S such that if $x \in S \setminus \bigcup_{p \in \mathcal{P}} M(p)$, then there exist $p, q \in \mathcal{P}$ with $q > p$ satisfying $x \in C(M(q), M(p))$. Here $C(M(q), M(p))$ describes the connecting orbits from $M(q)$ to $M(p)$ defined by $C(M(q), M(p)) := \{x \in S \mid \alpha(x) \subset M(q),\ \omega(x) \subset M(p)\}$ and $<$ denotes a partial order on $\mathcal{P}$, called an admissible order.

We add several remarks on Morse decompositions. First of all, since each $M(p)$ is an isolated invariant set, we can define the Conley index on it. Moreover, concerning the admissible order, the flow on S induces a natural ordering on the set $\mathcal{P}$ such that

$$\begin{aligned} p < q \iff & \exists\{\pi_0 (= p), \pi_1, \cdots, \pi_n (= q)\} \subset \mathcal{P} \\ & s.t.\ C(M(\pi_i), M(\pi_{i-1})) \neq \emptyset,\ i = 1, 2, \cdots, n. \end{aligned}$$

We denote this ordering by $<_F$. As is easily observed, any admissible orders $<$ are extensions of $<_F$, i.e. $p <_F q \Rightarrow p < q$.

Next, let us move on to the definition of the connection matrix on a Morse decomposition $\mathcal{M}(S, <)$.

DEFINITION 5. A connection matrix Δ_n is defined as a degree -1 linear map

$$\Delta_n : \bigoplus_{p\in\mathcal{P}} CH_n(M(p)) \to \bigoplus_{p\in\mathcal{P}} CH_{n-1}(M(p)),$$

which satisfies the following three conditions.

1. Δ_n is upper triangle, i.e. $\Delta_n(p,q) \neq 0 \Rightarrow q > p$
2. $\Delta_n\Delta_{n+1} = 0$
3. $\mathrm{Ker}\Delta_n/\mathrm{Im}\Delta_{n+1} \cong CH_n(S)$

Given a Morse decomposition $\mathcal{M}(S,<)$, we denote the set of connection matrices by $\mathcal{C}(<)$. Since an admissible order $<$ is an extension of $<_F$, $\mathcal{C}(<_F) \subset \mathcal{C}(<)$ can be proven.

The following two theorems [3] play a crucial role in the connection matrix theory.

THEOREM 6. $\mathcal{C}(<) \neq \emptyset$

THEOREM 7. Suppose $\Delta_n \in \mathcal{C}(<_F)$. If $\Delta_n(p,q) \neq 0$, then $C(M(q),M(p)) \neq \emptyset$. Therefore, since the existence of connection matrices are assured by Theorem 6, if we can somehow construct these matrices by using the algebraic restriction(Definition 5), then it becomes possible to detect the connecting orbits by Theorem 7.

In the followings, we explain the method to detect the connecting orbits between stationary solutions of the Swift-Hohenberg equation. The details, which include proofs, algorithms, and several numerical results, are summarized in [2]. Suppose $(k_0,\nu) = (0.62, 0.38)$ for example. Figure 3.1 describes the approximate bifurcation branches for equilibria of the Galerkin approximated Swift-Hohenberg equation at $k_0 = 0.62$. Due to the symmetry $u(x,t) \to -u(x,t)$ of (2.1), there seem to exist five

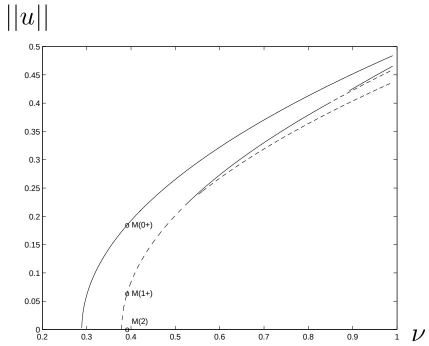

FIG. 3.1. *Bifurcation diagram at $k_0 = 0.62$.*

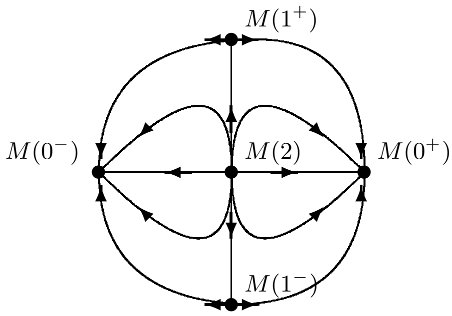

FIG. 3.2. *Semi-conjugacy*

stationary solutions $M(p)$, $p \in \mathcal{P} := \{0^\pm, 1^\pm, 2\}$ at $\nu = 0.38$ from Figure 3.1. Here $p^\pm$ imply the symmetry $M(p^+) = -M(p^-)$. By using the method explained in Section 2, this observation can be proven.

LEMMA 8. At $(k_0,\nu) = (0.62, 0.38)$, there uniquely exist stationary solutions in the neighborhood of the approximate equilibria described in Figure 3.1. Moreover, the Conley index for each stationary solution takes

$$CH_j(M(2)) \cong \begin{cases} \mathbb{Z}_2, & j = 2, \\ 0, & \text{otherwise} \end{cases}$$

$$CH_j(M(p^\pm)) \cong \begin{cases} \mathbb{Z}_2, & j = p, \\ 0, & \text{otherwise,} \end{cases} \quad p = 0, 1.$$

Note that we can not insist on the uniqueness of the solution by just using the method in the previous section. However, we can apply another numerical verification method developed in [10]. This technique is based on Banach's fixed point theorem and the application into the topological verification method enables us to prove the uniqueness of the stationary solutions[2].

Lemma 9. The set J described in Table 3.1 is a positive invariant lifting set and its Conley index takes

$$CH_j(J) \cong \begin{cases} \mathbb{Z}_2, & j = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

In addition, all the stationary solutions in J are those proven in Lemma 8.

Table 3.1
Positive invariant lifting set J

k	a_k^-	a_k^+
0	$-4.3380010295 \times 10^{-4}$	$4.3380010295 \times 10^{-4}$
1	$-3.4374821943 \times 10^{-3}$	$3.4374821943 \times 10^{-3}$
2	$-1.4440654070 \times 10^{-1}$	$1.4440654070 \times 10^{-1}$
3	$-4.5735140818 \times 10^{-5}$	$4.5735140819 \times 10^{-5}$
4	-1.0×10^{-4}	1.0×10^{-4}
5	-1.0×10^{-4}	1.0×10^{-4}
6	-1.0×10^{-4}	1.0×10^{-4}
$k \geq 7$	$-1.0/k^5$	$1.0/k^5$

The outline of the proof is the followings. First of all, the statement related to the positive invariant lifting set can be checked by the rigorous numerics as those performed in Section 2, since the vector field on the boundary of J is rigorously estimated. Obviously, since the exit set is empty, its Conley index should be (3.1). To complete the proof, it is sufficient to prove the nonexistence of the stationary solutions except for the regions where the existence of the stationary solutions are proven in Lemma 8. By adopting the nonexistence verification method[2] based on the mean value theorem, this is also verified by the rigorous numerics.

The existence of the Lyapunov function

$$F(u) = \int_0^{L_0} \left[\frac{1}{4}u^4 - \frac{\nu}{2}u^2 + \frac{1}{2}\Big(\left(1 + \partial_x^2\right) u\Big)^2 \right] dx,$$

Lemma 9, and the argument in [4] assure that $\mathcal{M}(S) = \{M(p) \mid p \in \mathcal{P}\}$ is a Morse decomposition in J. Hence, the connection matrices on $\mathcal{M}(S)$ clarifies the following theorem.

Theorem 10. At $(k_0, \nu) = (0.62, 0.38)$, the dynamics in J is semi-conjugate to the flow on the unit disk described in Figure 3.2.

The outline of the proof is the followings. We consider a chain complex such as

$$0 \longrightarrow \bigoplus_{p \in \mathcal{P}} CH_2(M(p)) \xrightarrow{\Delta_2} \bigoplus_{p \in \mathcal{P}} CH_1(M(p)) \xrightarrow{\Delta_1} \bigoplus_{p \in \mathcal{P}} CH_0(M(p)) \longrightarrow 0.$$

By Lemma 8, the symmetry $F(M(p^-)) = F(M(p^+))$, and the algebraic restriction (see Definition 5), the connection matrices Δ_2, Δ_1 become

$$\Delta_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, Theorem 7 shows the existence of connecting orbits described in Figure 3.2. For the construction of the semi-conjugate dynamics, see [2].

4. Efficient method for estimates of nonlinear terms. In this section, we discuss an efficient method to obtain estimates of nonlinear terms. As we discussed so far, the estimates of vector fields are inevitable for all computations. For this purpose, we need to especially estimate nonlinear terms,

$$\sum_{\substack{m_1+m_2+\cdots+m_p=j \\ m_i \in \mathbb{Z}}} a_{m_1} a_{m_2} \cdots a_{m_p},$$

where we assume the p-th nonlinearity for general cases. From the computational point of view, the computational cost for the finite sum

$$\sum_{\substack{m_1+m_2+\cdots+m_p=j \\ |m_i| \leq m}} a_{m_1} a_{m_2} \cdots a_{m_p}, \quad j = 0, 1, \ldots, m \tag{4.1}$$

grows rapidly with the order $O(m^p)$. Hence, it becomes difficult to use topological verification method for problems with large m and p. For example, localized patterns appearing in the quintic Swift-Hohenberg equation are one of the typical situations of this type[5]. Therefore, we try to reduce these computational costs in this section. The key idea comes from the pseudo spectral method with Fast Fourier Transform(FFT), which is well-known as one of the simulation methods for studying evolution equations.

Let us denote the discrete Fourier transform by

$$a_l = \mathcal{F}(u)\big|_l = \sum_{j=0}^{2m-1} u(x_j) e^{-ilk_0 x_j}, \tag{4.2}$$

$$u(x_j) = \mathcal{F}^{-1}(a)\big|_j = \frac{1}{2m} \sum_{l=-m+1}^{m} a_l e^{ilk_0 x_j}, \tag{4.3}$$

where $\{x_j = \frac{L_0}{2m} j\}$, $j = 0, 1, \cdots, 2m-1$ are grid points in the space $[0, L_0]$. The basic idea of the pseudo spectral method is the following. First, we pull back the Fourier coefficients $\{a_l\}$ to the original variable $\{u(x_j)\}$ by (4.3). Then we calculate the nonlinear term $\{u^p(x_j)\}$ at each point and again transform them into the Fourier region by (4.2). This argument can be checked in case of $p = 2$ (for the sake of simplicity) as,

$$c_l = \sum_{j=0}^{2m-1} u(x_j)^2 e^{-ilk_0 x_j} \tag{4.4}$$

$$= \frac{1}{(2m)^2}\sum_{j=0}^{2m-1}\left(\sum_{m_1=-m+1}^{m} a_{m_1}e^{im_1k_0x_j}\right)\left(\sum_{m_2=-m+1}^{m} a_{m_2}e^{im_2k_0x_j}\right)e^{-ilk_0x_j}$$

$$= \frac{1}{2m}\sum_{\substack{m_1+m_2=l\\ -m+1\le m_i\le m}} a_{m_1}a_{m_2} + \frac{1}{2m}\sum_{\substack{m_1+m_2=l\pm 2m\\ -m+1\le m_i\le m}} a_{m_1}a_{m_2}.$$

Note that the last term of (4.4) corresponds to the aliasing error. One of the methods to remove this error is the following[8]. Expand the size of Fourier coefficients from $2m$ to $2m\delta$ for some $\delta > 1$ as follows,

$$\begin{cases} a_j = 0, & \text{for } m+1 \le j \le \delta m \ \text{ and } \ -\delta m+1 \le j \le -m-1,\\ a_{-m} = a_m. \end{cases}$$

Then, the same calculation as above for the extended Fourier coefficients leads to

$$\begin{aligned} \hat{c}_l &= \sum_{j=0}^{2\delta m-1} u(x_j)^2 e^{-ilk_0x_j} \\ &= \frac{1}{2\delta m}\sum_{\substack{m_1+m_2=l\\ |m_i|\le m}} a_{m_1}a_{m_2} + \frac{1}{2\delta m}\sum_{\substack{m_1+m_2=l\pm 2\delta m\\ |m_i|\le m}} a_{m_1}a_{m_2}. \end{aligned} \tag{4.5}$$

Hence if we take $\delta > \frac{3}{2}$, then the aliasing term can be eliminated and the finite summation (4.1) for $p = 2$ is calculated by

$$\sum_{\substack{m_1+m_2=l\\ |m_i|\le m}} a_{m_1}a_{m_2} = 2m\delta\hat{c}_l.$$

Note that by using FFT equipped with interval arithmetic we obtain the rigorous estimates of (4.1) quite efficiently.

We conclude this section by showing the several results related to the localized patterns appearing in the quintic Swift-Hohenberg equation[5]. Figure 4.1 is a bifurcation diagram for approximate equilibria for the quintic Swift-Hohenberg equation:

$$\begin{aligned} u_t &= \left\{\nu - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right\}u + \mu u^3 - u^5, \\ u(x,t) &= u(x+L_0,t), \quad u \in L^2(0,L_0), \end{aligned} \tag{4.6}$$

where ν and μ are parameters. In the diagram, each layer on the snaky bifurcation branch is labeled as U_k, S_k corresponding to its stability(U_k: unstable, S_k: stable). The wave profiles on these layers are shown in Figure 4.2. As is easily observed, the wave profiles on these layers are localized patterns. This fact implies that the number m in the Fourier expansion should be taken large enough to perform the verification. The value of m in our verification is shown at the end of this section. In order to apply the FFT technique, we prepare the following lemma.

Lemma 11. The constant δ to remove aliasing errors should satisfy $\delta > 2$ for $p = 3$ and $\delta > 3$ for $p = 5$, respectively.

Let us denote the stationary solution corresponding to the equilibrium on the lower layers of the snaky branch in Figure 4.1 by

$$u(x; k_0, \nu, b) = \sum_{|j|\le m} a_j \cos(jk_0x),$$

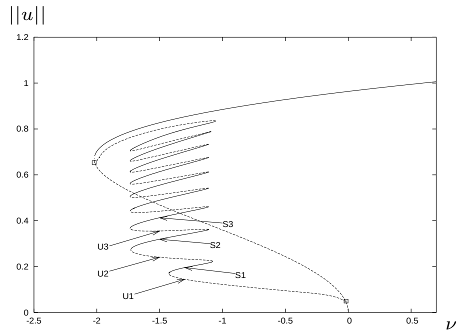

Fig. 4.1. *Bifurcation diagram at $\mu = 3.0$ ($k_0 = 0.1$).*

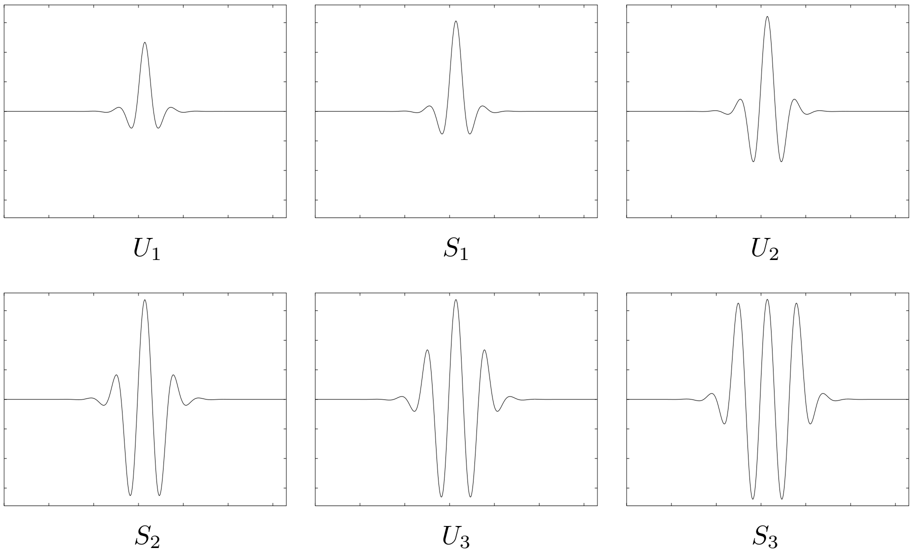

Fig. 4.2. *Profiles of equilibria on the layers U_k, S_k, $k = 1, 2, 3$, ($\nu = -1.3$).*

where the set of the Fourier coefficients $\{a_j\}$ corresponds to the approximate equilibrium on each layer $b = U_k, S_k,\ k = 1, 2, 3$, at the parameter value ν. Then, we obtain the following theorems by the topological verification method.

Theorem 12. Let $\nu = -1.3$, $\mu = 3.0$ and $k_0 = 0.1$. Then, around each approximate solution $u(x; k_0, \nu, b)$, $b = U_k, S_k,\ k = 1, 2, 3$, there exists a stationary solution $u_*(x; k_0, \nu, b)$ of the quintic Swift-Hohenberg equation (4.6) such that

$$||u_*(\cdot; k_0, \nu, U_1) - u(\cdot; k_0, \nu, U_1)||_{L^2} \le 1.04077019 \times 10^{-8}$$
$$||u_*(\cdot; k_0, \nu, S_1) - u(\cdot; k_0, \nu, S_1)||_{L^2} \le 1.57739803 \times 10^{-8}$$
$$||u_*(\cdot; k_0, \nu, U_2) - u(\cdot; k_0, \nu, U_2)||_{L^2} \le 2.44819377 \times 10^{-8}$$
$$||u_*(\cdot; k_0, \nu, S_2) - u(\cdot; k_0, \nu, S_2)||_{L^2} \le 4.31155312 \times 10^{-8}$$
$$||u_*(\cdot; k_0, \nu, U_3) - u(\cdot; k_0, \nu, U_3)||_{L^2} \le 2.83246161 \times 10^{-9}$$
$$||u_*(\cdot; k_0, \nu, S_3) - u(\cdot; k_0, \nu, S_3)||_{L^2} \le 7.47772691 \times 10^{-9}$$

Theorem 13. Let $\nu = -1.5$, $\mu = 3.0$ and $k_0 = 0.1$. Then, around each approximate solution $u(x;k_0,\nu,b)$, $b=U_k,\ S_k,\ k=2,3$, there exists a stationary solution $u_*(x;k_0,\nu,b)$ of the quintic Swift-Hohenberg equation (4.6) such that

$$||u_*(\cdot;k_0,\nu,U_2)-u(\cdot;k_0,\nu,U_2)||_{L^2} \leq 4.47782900\times 10^{-8}$$
$$||u_*(\cdot;k_0,\nu,S_2)-u(\cdot;k_0,\nu,S_2)||_{L^2} \leq 4.57533187\times 10^{-8}$$
$$||u_*(\cdot;k_0,\nu,U_3)-u(\cdot;k_0,\nu,U_3)||_{L^2} \leq 4.20841075\times 10^{-8}$$
$$||u_*(\cdot;k_0,\nu,S_3)-u(\cdot;k_0,\nu,S_3)||_{L^2} \leq 6.82912523\times 10^{-9}$$

It should be remarked that we set the power decay property (2.5) as $c=1.0$ and $s=5$ for all the verifications of the above theorems. Moreover, the dimension for the finite part is chosen as $m=256$ for $U_k,S_k,\ k=1,2$, and $m=512$ for U_3,S_3 in Theorem 12. In Theorem 13, $m=256$ for U_2,S_2,U_3 and $m=512$ for S_3.

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REFERENCES

[1] C. Conley. Isolated Invariant Sets and the Morse Index, CBMS Lecture Notes **38**, A.M.S. Providence, R.I. (1978).
[2] S. Day, Y. Hiraoka, K. Mischaikow, and T. Ogawa. Rigorous Numerics for Global Dynamics: a study of the Swift-Hohenberg equation, in press(*SIAM J. Appl. Dynm. Sys.*).
[3] R. Franzosa. The connection matrix theory for Morse decompositions, *Trans. Amer. Math. Soc.* **311** 561 (1989).
[4] J. K. Hale. Asymptotic Behavior of Dissipative Systems, **25** A.M.S. Providence, R.I. (1988).
[5] Y. Hiraoka and T. Ogawa. Rigorous numerics for localized patterns to the quintic Swift-Hohenberg equation, preprint.
[6] K. Mischaikow and M. Mrozek. Conley index, Handbook of dynamical systems, Vol. 2, North-Holland, 393 (2002).
[7] M. T. Nakao. A numerical verification method for the existence of weak solutions for nonlinear boundary value problems, *J. Math. Anal. Appl.* **164**, 489 (1992).
[8] S. Orszag. Numerical Simulation of Incompressible Flows within Simple Boundaries. I. Galerkin (Spectral) Representations, *Stud. Appl. Math.* **50**, 293 (1971).
[9] D. Salamon. Connected simple systems and the Conley index of isolated invariant sets, *Trans. Amer. Math. Soc.* **291**, 1 (1985).
[10] N. Yamamoto. A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed-point theorem, *SIAM J. Numer. Anal.* **35** 2004 (1998).
[11] P. Zgliczyński and K. Mischaikow. Rigorous Numerics for Partial Differential Equations: The Kuramoto-Sivashinsky Equation, *Found. Comput. Math.* **1**, 255 (2001).