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# GLOBAL EXISTENCE OF SOLUTIONS FOR A FREE BOUNDARY PROBLEM OF HYPERBOLIC TYPE WITH NON CONSTANT ADHESION

KAZUAKI NAKANE[1] AND TOMOKO SHINOHARA[2]

**Abstract.** A free boundary problem which arises from the physical model " Peel a thin film from a domain" is treated. The behavior of the peeling front is governed by the hyperbolic equation. If we suppose the effect $Q$ from the peeling front to Lagrangian is constant, the global solution have been constructed by pasting the local solutions inductively. In this note, the case where $Q$ is a function of the space is treated. Because the effect form peeling front is supposed to depend on the situation of the domain. If we impose a regularity condition on $Q$, the sufficient condition for the global existence of the solution is given.

**Key words.** free boundary, hyperbolic equation, variational problem

**AMS subject classifications.** 35R35, 35L70

**1. Introduction.** Let us consider the following one-dimensional free boundary problem

$$(P) \quad \begin{cases} u_{xx} - u_{tt} = 0 & \text{in} \quad (0,\infty)\times\{t>0\}\cap\{u>0\}, \\ u_x^2 - u_t^2 = Q^2 & \text{on} \quad (0,\infty)\times\{t>0\}\cap\partial\{u>0\}, \end{cases}$$

with the initial conditions

$$(I) \quad \begin{cases} u(x,0) = e(x) & \text{in} \quad (-l_0, 0), \\ u_t(x,0) = g(x) & \text{in} \quad (-l_0, 0), \end{cases}$$

and the boundary condition

$$(B) \quad u(-l_0, t) = f(t) \qquad \text{for} \quad t \ge 0,$$

where $e(x)$, $g(x)$, $f(t)$ and $Q$ are given functions, and $l_0$ is a positive constant.

This problem arises from the following variational problem which is related to a physical model "Peel a thin film from a domain $\Omega$ "(cf. [7] ) The shape of a film is described by the graph of a function $u : \Omega \to \mathbf{R}$. Find a stationary point of the functional

$$(1.1) \qquad J(u) := \int_0^{T^*} \int_\Omega \left( \frac{\tau}{2}|\nabla u|^2 - \frac{\rho}{2}(D_t u)^2 \chi_{u>0} + \frac{Q^2}{2\tau}\chi_{u>0} \right) dxdt \qquad u \in \mathrm{K},$$

where $\Omega$ is a domain in $\mathbf{R}^n$, $T^*$ is a positive constant, $\chi_{u>0}$ is a characteristic function of the set $\{(x,t) \in \Omega\times(0,T^*); u(x,t) > 0\}$ and K is a suitable function space. Here the

[1]Department of Engineering, Osaka Institute of Technology, Asahi-ku, Osaka 535–8585, Japan.
The author is partially supported by Grant-in-Aid for Young Scientists (B)(No. 14740090) of Japan Society for Promotion of Science.
[2]Tokyo Metropolitan College of Technology, 1-10-40 Higashi-ooi, Shinagawa, Tokyo 140-0011, Japan.

constants $\tau$, $\rho$ are the tension and the line density, and $Q$ is the adhesion. To approach this problem, we assume that a stationary point is sufficiently smooth. Then we can derive (1.2) and (1.3) as the Euler-Lagrange equations from the functional (1.1) (cf. [1], [2] and [4]),

$$\tau\Delta u - \rho u_{tt} = 0 \qquad \text{in} \quad \Omega \times \{t > 0\} \cap \{u > 0\}, \tag{1.2}$$

$$\frac{\tau}{2}|\nabla u|^2 - \frac{\rho}{2}u_t^2 = \frac{Q^2}{2\tau} \qquad \text{on} \quad \Omega \times \{t > 0\} \cap \partial\{u > 0\}. \tag{1.3}$$

In this article, as a first step, a one-dimensional problem will be analyzed. By using some change of variable $t$, $\tau = 1$ and $\rho = 1$ can be assumed. Therefore, we will consider the problem $(P)$, $(I)$ and $(B)$.

The initial condition $(I)$ implies that a thin film has been already peeled from the plate on the interval $(-l_0, 0)$ and the boundary condition $(B)$ corresponds to the situation in which the edge of film is lifted up by $f(t)$. Kikuchi and Omata [4] showed the existence of time-local solutions under the several conditions which were imposed on the functions $e(x)$, $g(x)$ and $f(t)$. On the global existence of solutions, however, they have not stated.

In the case of $Q$ is constant, numerical experiments were carried out in [3]. A sufficient condition for the global existence was given in [6]. That is to say, the solution can be constructed by pasting the local solutions inductively, if $f' \geq 0$, the solution can be extended to any $T > 0$. And the same time, by using the expression of the solution of this way, a well-posedness can be shown and the periodic solutions are constructed.

In this note, the case where $Q$ is a function of the space is treated, namely, it is given on the line $\{(x,t); t = 0\}$. Because the effect $Q$ from the peeling front to the Lagrangian is supposed to depend on the situation of the domain. If we impose a condition (A.5) on $Q$, then the global solution can be constructed.

**2. Construction of solutions.** To prove our problem, two variables $\xi$ and $\eta$ are introduced by

$$t = (\xi + \eta)/2,$$
$$x = (\xi - \eta)/2.$$

Since the initial values $e$ and $g$ are given on the line $\{(x,t); t = 0\}$, they are given on the line $\{(\xi,\eta); \xi + \eta = 0\}$. We regard them as the functions of $\xi$ and rewrite them $e$ and $g$ again. Similarly, the boundary value $f$ and the adhesion $Q$ are given on the line $\{(\xi,\eta); \xi - \eta + 2l_0 = 0\}$ and $\{(x,t); t = 0\}$, respectively. They are functions of $\eta$. Therefore $(P)$, $(I)$ and $(B)$ are transformed into

$$(P') \quad \begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{u > 0\}, \\ -4u_\xi u_\eta = Q^2(\eta) & \text{on} \quad \partial\{u > 0\}, \end{cases}$$

$$(I') \quad \begin{cases} u(\xi, -\xi) = e(\xi) & \text{in} \quad (-l_0, 0), \\ u_\eta(\xi, -\xi) + u_\xi(\xi, -\xi) = g(\xi) & \text{in} \quad (-l_0, 0), \end{cases}$$

$$(B') \quad u(\eta - 2l_0, \eta) = f(\eta - l_0) \qquad \text{in} \quad [l_0, \infty).$$

By taking these equations into considerations, we treat the following problem:

**Problem 2.1** *Let $T$ be a positive constant. Find a pair of functions $u \in C^0(\{(\xi,\eta);\xi \geq \eta - 2l_0, \xi \geq -\eta\})$ and $l \in C^0([0,T)) \cap C^1((0,T))$ which satisfies $(P')$, $(I')$ and $(B')$ for $\eta < T$ and*
(i) $l(0) = 0$,
(ii) $u \in C^2(\{(\xi,\eta);\eta - 2l_0 < \xi < l(\eta), \xi > -\eta\}) \cap C^1(\{(\xi,\eta);\eta - 2l_0 < \xi \leq l(\eta), \xi \geq -\eta\})$,
(iii) $u > 0$ *in* $\{(\xi,\eta);\eta - 2l_0 \leq \xi < l(\eta), \xi > -\eta\}$,
(iv) $u(\xi,\eta) = 0$ *in* $\{(\xi,\eta);\xi \geq l(\eta)\} \cup \{(\xi,\eta);\xi \geq -\eta, \eta < 0\}$.

**Assumption 2.1** *The functions $f(\eta - l_0) \in C^2([l_0,\infty))$, $e(\xi) \in C^2([-l_0,0])$, $g(\xi) \in C^1([-l_0,0])$ and $Q(\eta) \in C^0([0,\infty))$ satisfy*

$$\text{(A.0)} \quad \begin{cases} e(\xi) > 0 & \text{in} \quad (-l_0,0), \\ g(\xi) > 0 & \text{in} \quad (-l_0,0), \end{cases}$$

$$\text{(A.1)} \quad \begin{cases} f(0) = e(-l_0) > e(0) = 0, \\ f'(0) = g(-l_0), \\ f''(0) = e''(-l_0), \end{cases}$$

$$\text{(A.2)} \quad \begin{cases} e(0) = 0, \\ e'(0)^2 - g(0)^2 = Q^2(0), \\ -Q^3(0)\{2Q'(0)(-e'(0) + g(0)) - Q(0)(e''(0) - g'(0))\} \\ = (-e'(0) + g(0))^4(e''(0) + g''(0)), \end{cases}$$

$$\text{(A.3)} \quad \begin{cases} e'(\xi) < g(\xi) & \text{for} \quad (-l_0 \leq \xi \leq 0), \\ f'(\xi + l_0) - (e'(\xi) + g(\xi))/2 > 0 & \text{for} \quad (-l_0 \leq \xi \leq 0), \end{cases}$$

$$\text{(A.4)} \quad f'(\eta - l_0) \geq 0 \quad \text{for} \quad \eta \in [l_0, \infty).$$

(A.5) $Q$ is *locally monotone function, i.e.* for any $\eta_0$
there is a constant $\delta(\eta_0) > 0$ such that $Q(\eta_1) \leq Q(\eta_2)$ or
$Q(\eta_1) \geq Q(\eta_2)$ for all $\eta_1,\ \eta_2 \in [\eta_0 - \delta(\eta_0), \eta_0]$ with $\eta_1 \leq \eta_2$.

**Remark 2.1** (i) (A.1) *is a compatible condition on the lifting edge and* (A.2) *is a condition on the peeling front at $t = 0$. From our physical model, it is easy to see that the singularities start from these points and they propagate along the lines which run parallel to the $(\eta,\xi)$-axes. To show the regularity of the solution, we need to impose some conditions on them. If the first equations of* (A.1) *and* (A.2) *are satisfied, we can show the solution is continuous. By using direct calculation, the second one and third one guarantee $C^1$ and $C^2$ regularity of the solution, respectively.*
(ii) (A.3) *is assumed by the technical reason. We may consider* (A.3) *is consistent with the domain of dependence, however, we do not know the relation between them.*
(iii) (A.4) and (A.5) *are needed to construct global solutions. Even if $Q$ belongs to $C^k$ class for any $k > 0$, there is an example such that it does not enjoy* (A.5)($x^k \sin \dfrac{1}{x}$ *is not locally monotone at* 0). *However,* (A.5) *is satisfied by the function which has no point where the signature of the $Q'$ changes infinity many time in its neighborhood.*

The following procedure is originally given in [4]. To confirm the global existence of the solution, we will modify its proof and introduce an iteration method.

**Lemma 2.1** *Let $c$ be a positive number and $I = [0, c)$ an interval on the axis $\eta$. Let $Q(\eta) \in C^0([0,\infty))$ and $\lambda(\eta) \in C^2(I)$ be a function which satisfies*
(i)$\lambda(0) = 0$,
(ii)$\lambda'(\eta) > 0$ on $\eta \in I$.
*Then there exists a unique pair of functions $u(\xi,\eta) \in C^2(\{(\xi,\eta); \eta \in I, 0 < \xi < l(\eta)\}) \cap C^1(\{(\xi,\eta); \eta \in I, 0 \le \xi \le l(\eta)\})$ and $l(\eta) \in C^2(I)$ such that*

$$(PL)\begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{(\xi,\eta); 0 < \xi < l(\eta), \eta \in I\}, \\ -4u_\xi u_\eta = Q^2(\eta) & \text{on} \quad (l(\eta),\eta), \quad \eta \in I, \\ u(l(\eta),\eta) = 0 & \text{on} \quad \eta \in I, \\ u(0,\eta) = \lambda(\eta) & \text{on} \quad \eta \in I. \end{cases}$$

*Proof.* At first, let us define the functions $\psi$ and $l$ by

$$\begin{aligned} \psi(\eta) &= \lambda(\eta), \\ l(\eta) &= 4\int_0^\eta \frac{\psi'(s)^2}{Q^2(s)} ds. \end{aligned}$$

Because $\psi'$ is positive, $l^{-1}(\xi)$ exists in $[0, l(c))$. The function $\phi$ is defined by

$$\phi(\xi) = -\psi(l^{-1}(\xi)).$$

Then we can see that $u(\xi,\eta) = \phi(\xi) + \psi(\eta)$ and $l(\eta)$ are desired functions.

Nextly, we show the uniqueness of the solution. Suppose that there exist another functions $(\tilde{u}(\xi,\eta), \tilde{l}(\eta))$ satisfying $(PL)$ which has the form $\tilde{u}(\xi,\eta) = \tilde{\phi}(\xi) + \tilde{\psi}(\eta)$. It follows that

$$\tilde{u}(\tilde{l}(\eta),\eta) = 0. \tag{2.1}$$

From the fact $-4\tilde{u}_\xi \tilde{u}_\eta = Q^2$ and by differentiating the both sides of (2.1) with respect to $\eta$, we have

$$\tilde{l}'(\eta) = \frac{4}{Q^2(\eta)} \tilde{\psi}'(\eta)^2.$$

Since the free boundary starts at origin, we obtain

$$\tilde{l}(\eta) = 4\int_0^\eta \frac{\tilde{\psi}'(s)^2}{Q^2(s)} ds.$$

Because of

$$\tilde{u}(0,\eta) = \tilde{\phi}(0) + \tilde{\psi}(\eta) = \lambda(\eta),$$

it follows

$$\tilde{\psi}'(\eta) = \lambda'(\eta).$$

It implies that $\tilde{l}$ is equal to $l$. Hence, from Goursat's theorem, $(\tilde{u}(\xi,\eta), \tilde{l}(\eta))$ coincides with $(u(\xi,\eta), l(\eta))$, it is a contradiction proving our assertion. ∎

**Main Theorem** *For any $T > 0$, there exists a unique solution to Problem 2.1.*

*Proof.* Firstly, let $\phi_0$ and $\psi_1$ be functions such that

$$\begin{cases} \phi_0(\xi) = \dfrac{1}{2}\left(e(\xi) + \displaystyle\int_0^{\xi} g(s)ds\right) & \text{for} \quad -\lambda_1 \le \xi \le 0, \\ \psi_1(\eta) = \dfrac{1}{2}\left(e(-\eta) + \displaystyle\int_{-\eta}^{0} g(s)ds\right) & \text{for} \quad 0 \le \eta < \lambda_1, \end{cases}$$

where $\lambda_1 = l_0$. Evidently, $u(\xi,\eta) = \phi_0(\xi) + \psi_1(\eta)$ is a unique solution to the initial value problem in $\{(\xi,\eta); -l_0 < -\eta < \xi < 0\}$. Let us define the free boundary $l_1$ by

$$l_1(\eta) = 4\int_0^{\eta} \frac{\psi_1'(s)^2}{Q^2(s)} ds.$$

By (A.3), $\psi_1'(\eta)$ is positive for $0 \le \eta < \lambda_1$. Therefore there exists $l_1^{-1}(\xi)$ for $0 \le \xi < l_1(\lambda_1)$ and we can define $\phi_1$ by

$$\phi_1(\xi) = -\psi_1(l_1^{-1}(\xi)) \qquad \text{for} \quad 0 \le \xi < l_1(\lambda_1).$$

By using these functions, we define the functions $u$ and $l$ by

$$u(\xi,\eta) = \begin{cases} \phi_0(\xi) + \psi_1(\eta) & \text{on} \quad D_{0,1}, \\ \phi_1(\xi) + \psi_1(\eta) & \text{on} \quad D_{1,1}, \\ 0 & \text{on} \quad D_{e,1}, \end{cases}$$

$$l(\eta) = l_1(\eta) \qquad \text{for} \quad \{\eta; 0 \le \eta < \lambda_1\},$$

where

$$\begin{aligned} D_{0,1} &= \{(\xi,\eta); \xi \ge -\eta, -l_0 \le \xi < 0, 0 \le \eta < \lambda_1\}, \\ D_{1,1} &= \{(\xi,\eta); 0 \le \xi < l_1(\eta), 0 \le \eta < \lambda_1\}, \\ D_{e,1} &= \{(\xi,\eta); \xi \ge l_1(\eta), 0 \le \eta < \lambda_1, \}. \end{aligned}$$

The regularity on the joint line $\xi = 0$ of $\phi_0$ and $\phi_1$ is guaranteed by (A.2). By Lemma 2.1, $(u(\xi,\eta),\, l(\eta))$ is a unique solution to Problem 2.1 on $D_{0,1} \cup D_{1,1} \cup D_{e,1}$.

Nextly, we define $\psi_2$ by

$$\psi_2(\eta) = \begin{cases} f(\eta - l_0) - \phi_0(\eta - 2l_0) & \text{for} \quad \lambda_1 \le \eta < 2\lambda_1, \\ f(\eta - l_0) - \phi_1(\eta - 2l_0) & \text{for} \quad 2\lambda_1 \le \eta < \lambda_2, \end{cases}$$

where $\lambda_2 = l_1(\lambda_1) + 2l_0$. Since $\phi_0$ and $\phi_1$ are connected smoothly, $\psi_2(\eta)$ is of $C^2$-class. Because the free boundary starts from $l_1(\lambda_1)$, $l_2$ is defined by

$$l_2(\eta) = 4\int_{\lambda_1}^{\eta} \frac{\psi_2'(s)^2}{Q^2(s)} ds + l_1(\lambda_1) \qquad \text{for} \quad \lambda_1 \le \eta < \lambda_2.$$

Obviously, $l_2$ is of $C^2$-class. By (A.3) and (A.4), $\psi_2'(\eta) > 0$ holds on $[\lambda_1, \lambda_2)$. Then there exists $l_2^{-1}(\xi)$ for $l_2(\lambda_1) \le \xi < l_2(\lambda_2)$. Hence we can define

$$\phi_2(\xi) = -\psi_2(l_2^{-1}(\xi)) \quad \text{for} \quad l_2(\lambda_1) \le \xi < l_2(\lambda_2).$$

The functions $(u(\xi,\eta), l(\eta))$ are extended as the following

$$u(\xi,\eta)=\begin{cases}\phi_0(\xi)+\psi_2(\eta) & \text{on} \quad D_{0,2},\\ \phi_1(\xi)+\psi_2(\eta) & \text{on} \quad D_{1,2},\\ \phi_2(\xi)+\psi_2(\eta) & \text{on} \quad D_{2,2},\\ \quad 0 & \end{cases}$$

$$l(\eta)=l_2(\eta) \qquad \text{on} \quad \{\eta; \lambda_1 \le \eta < \lambda_2\},$$

where

$$\begin{aligned} D_{0,2} &= \{(\xi,\eta); \xi \ge \eta-2l_0, -l_0 \le \xi < 0, \quad \lambda_1 \le \eta < 2l_0\},\\ D_{1,2} &= \{(\xi,\eta); \xi \ge \eta-2l_0, 0 \le \xi < l_1(\lambda_1), \quad \lambda_1 \le \eta < \lambda_2\},\\ D_{2,2} &= \{(\xi,\eta); l_1(\lambda_1) \le \xi < l_2(\eta), \quad l_1 \le \eta < \lambda_2\},\\ D_{e,2} &= \{(\xi,\eta); \xi \ge l_2(\eta), \lambda_1 \le \eta < \lambda_2\}. \end{aligned}$$

It follows from (A.1) that $\psi_1$ and $\psi_2$, concurrently, $\phi_1$ and $\phi_2$ are connected smoothly on the line $\xi=\lambda_2-2l_0$. By applying Lemma 2.1 again, we can see the pair of functions $(u(\xi,\eta), l(\eta))$ is a unique solution to Problem 2.1.

Inductively, for $j \ge 3$, we define the functions $\psi_j$, $l_j$ and $\phi_j$ by

$$\begin{aligned} &\psi_j(\eta)=f(\eta-l_0)-\phi_{j-1}(\eta-2l_0) && \text{for} \quad \lambda_{j-1}\le\eta<\lambda_j,\\ &l_j(\eta)=4\int_{\lambda_{j-1}}^{\eta}\frac{\psi_j'(s)^2}{Q^2(s)}ds+l_{j-1}(\lambda_{j-1}) && \text{for} \quad \lambda_{j-1}\le\eta<\lambda_j,\\ &\phi_j(\xi)=-\psi_j(l_j^{-1}(\xi)) && \text{for} \quad l_j(\lambda_{j-1})\le\xi<l_j(\lambda_j), \end{aligned}$$

where

$$\lambda_j=l_{j-1}(\lambda_{j-1})+2l_0.$$

Suppose that it has been already shown that $\psi_{j-1}(\eta)$, $l_{j-1}(\eta)$ and $\phi_{j-1}(\xi)$ are well-defined and of $C^2$-class, and $\psi_{j-1}'(\eta)>0$ on $\lambda_{j-2}\le\eta<\lambda_{j-1}$. In addition, it is supposed that $\phi_{j-2}$ and $\phi_{j-1}$ are connected smoothly on the line $\xi=\lambda_{j-1}-2l_0$. Since

$$\begin{aligned} \lambda_j-\lambda_{j-1} &= 4\int_{\lambda_{j-2}}^{\lambda_{j-1}}\frac{\psi_{j-1}'(s)^2}{Q^2(s)}ds \\ &\ge 4\left(\min_{\lambda_{j-2}\le\eta<\lambda_{j-1}}\frac{\psi_{j-1}'(\eta)}{Q(\eta)}\right)^2(\lambda_{j-1}-\lambda_{j-2}), \end{aligned} \tag{2.2}$$

holds, we have $\lambda_j>\lambda_{j-1}$. Here we remark that $l_j(\lambda_{j-1})=l_{j-1}(\lambda_{j-1})$. It can be seen that $\psi_j$ and $l_j$ are well-defined and of $C^2$-class, immediately. We have

$$\begin{aligned} \psi_j'(\eta) &= f'(\eta-l_0)-\phi_{j-1}'(\eta-2l_0)\\ &= f'(\eta-l_0)+\frac{1}{4}\frac{Q^2(l_{j-1}^{-1}(\eta-2l_0))}{\psi_{j-1}'(l_{j-1}^{-1}(\eta-2l_0))}, \end{aligned}$$

it follows $\psi_j'(\eta)>0$ on $\lambda_{j-1}\le\eta<\lambda_j$ by (A.4) and induction assumption. Therefore, there exists $l_j^{-1}$ for $l_j(\lambda_{j-1})\le\xi<l_j(\lambda_j)$, and it implies that $\phi_j$ is well-defined and

of $C^2$-class. From the induction assumption, $\psi_{j-1}$ and $\psi_j$ are connected smoothly on the line $\eta = \lambda_{j-1}$. Therefore $\phi_{j-1}$ and $\phi_j$ are connected smoothly on the line $\xi = \lambda_j - 2l_0$. By combining these functions, the pair of functions $(u(\xi,\eta), l(\eta))$ can be extended as the following

$$u(\xi,\eta) = \begin{cases} \phi_{j-1}(\xi) + \psi_j(\eta) & \text{on} \quad D_{j-1,j}, \\ \phi_j(\xi) + \psi_j(\eta) & \text{on} \quad D_{j,j}, \\ 0 & \text{on} \quad D_{e,j}, \end{cases}$$

$$l(\eta) = l_j(\eta) \qquad \text{for} \quad \{\eta; \lambda_{j-1} \le \eta < \lambda_j\},$$

where

$$\begin{aligned} D_{j-1,j} &= \{(\xi,\eta); \xi \ge \eta - 2l_0, \lambda_{j-1} - 2l_0 \le \xi < l_{j-1}(\lambda_{j-1}), \lambda_{j-1} \le \eta < \lambda_j\}, \\ D_{j,j} &= \{(\xi,\eta); l_{j-1}(\lambda_{j-1}) \le \xi < l_j(\eta), \lambda_{j-1} \le \eta < \lambda_j\}, \\ D_{e,j} &= \{(\xi,\eta); \xi \ge l_j(\eta), \lambda_{j-1} \le \eta < \lambda_j\}. \end{aligned}$$

By the same argument on $j = 2$, $(u(\xi,\eta), l(\eta))$ is a unique solution to Problem 2.1.

Finally, we shall show the sequence $\{\lambda_j\}_{j=1}^{\infty}$ goes to infinity. Once we show this, for any $T > 0$, a pair of functions $(u, l)$ which is a solution to Problem 2.1 can be constructed. To this end, *assume the contrary.* Then, $\{\lambda_j\}_{j=1}^{\infty}$ converges to some constant $\lambda_\infty < \infty$. Then, it follows from the definitions

$$\begin{aligned} &\lambda_{j+2} - \lambda_{j+1} \\ &= 4\int_{\lambda_j}^{\lambda_{j+1}} \frac{\psi_{j+1}'(s)^2}{Q^2(s)} ds \\ &= 4\int_{\lambda_j}^{\lambda_{j+1}} \frac{1}{Q^2(s)} \left\{ f'(s - l_0) + \frac{Q^2\left(l_j^{-1}(s - 2l_0)\right)}{4\psi_j'\left(l_j^{-1}(s - 2l_0)\right)} \right\}^2 ds \\ &\ge 4\int_{\lambda_j}^{\lambda_{j+1}} \frac{1}{Q^2(s)} \left\{ \frac{Q^2\left(l_j^{-1}(s - 2l_0)\right)}{4\psi_j\left(l_j^{-1}(s - 2l_0)\right)} \right\}^2 ds. \end{aligned}$$

Using the change of variable $y = l_j^{-1}(s - 2l_0)$, we have

$$\lambda_{j+2} - \lambda_{j+1} \ge \int_{\lambda_{j-1}}^{\lambda_j} \frac{Q^2(y)}{Q^2\left(l_j(y + 2l_0)\right)} dy.$$

From (A.5) there exists a constant $\delta$ such that $Q$ is a monotone function on $[\lambda_\infty - \delta, \lambda_\infty)$. Let $\lambda_{j_0} \in \{\lambda_j\}_{j=1}^{\infty}$ be the lowest number which is in $[\lambda_\infty - \delta, \lambda_\infty)$. Here, we have two cases to consider.

**(Case 1)** $Q$ is decreasing on $[\lambda_\infty - \delta, \lambda_\infty)$.

It holds

$$\frac{Q^2(\bar\lambda)}{Q^2\left(l_j(\bar\lambda) + 2l_0\right)} \ge 1 \quad \text{for } \bar\lambda \in [\lambda_{j_0}, \lambda_\infty).$$

Therefore, we obtain

$$\lambda_{j+2} - \lambda_{j+1} \ge \int_{\lambda_{j-1}}^{\lambda_j} dy = \lambda_j - \lambda_{j-1}, \text{ and } \lim_{j\to\infty} \lambda_j = \infty.$$

It contradicts our assumption. In this case the proof completes.
**(Case 2)** $Q$ is increasing on $[\lambda_\infty - \delta, \lambda_\infty)$.
From mean value theorem for integral, there exists a sequence $\bar{\lambda}_j \in [\lambda_{j-1}, \lambda_j)$ such that

$$\lambda_{j+2} - \lambda_{j+1} \geq \int_{\lambda_{j-1}}^{\lambda_j} \frac{Q^2(y)}{Q^2(l_j(y) + 2l_0)} dy = (\lambda_j - \lambda_{j-1}) q_j, \tag{2.3}$$

where $q_j = \dfrac{Q^2(\bar{\lambda}_j)}{Q^2(l_j(\bar{\lambda}_j) + 2l_0)}$. By using (2.3) inductively, it follows that

$$\lambda_{j_0+2j} - \lambda_{j_0+2j-1} \geq (\lambda_{j_0+2} - \lambda_{j_0+1}) \sum_{k=1}^{j-1} q_{j_0+2} \cdots q_{j_0+2k},$$

$$\lambda_{j_0+2j-1} - \lambda_{j_0+2j-2} \geq (\lambda_{j_0+1} - \lambda_{j_0}) \sum_{k=1}^{j-1} q_{j_0+2} \cdots q_{j_0+2k-1}.$$

Hence, we have

$$\begin{aligned} &\lambda_\infty - \lambda_{j_0} \\ &= \lim_{j\to\infty} \{(\lambda_{j_0+1} - \lambda_{j_0}) + \cdots + (\lambda_{j_0+2j} - \lambda_{j_0+2j-1})\} \\ &\geq (\lambda_{j_0+1} - \lambda_{j_0}) \sum_{k=1}^{j-1} q_{j_0+1} \cdots q_{j_0+2k-1} \\ &\quad + (\lambda_{j_0+2} - \lambda_{j_0+1}) \sum_{k=1}^{j-1} q_{j_0+2} \cdots q_{j_0+2k}. \end{aligned} \tag{2.4}$$

From the assumption $\lambda_\infty < \infty$, the right hand side of (2.4) converges. Therefore, it implies that

$$\lim_{j\to\infty} q_{j_0+1} \cdots q_{j_0+2j-1} = 0 \text{ and } \lim_{j\to\infty} q_{j_0+2} \cdots q_{j_0+2j} = 0. \tag{2.5}$$

Let $\{M_j\}$ and $\{m_j\}$ be recurrences such that

$$M_j = \max_{\eta \in [\lambda_{j-1}, \lambda_j]} Q(\eta) \quad \text{and} \quad m_j = \min_{\eta \in [\lambda_{j-1}, \lambda_j]} Q(\eta).$$

Then, because of $Q^2(l_j(\bar{\lambda}_j) + 2l_0) = \dfrac{Q^2(\bar{\lambda}_j)}{q_j}$, it holds $M_{j+1}^2 \geq \dfrac{m_j^2}{q_j}$. Since $Q$ is increasing, $m_{j+1}^2 = M_j^2$ holds. Then we obtain

$$\begin{aligned} m_{j+1}^2 &= M_j^2 \geq \frac{m_{j-1}^2}{q_{j-1}} \\ &= \frac{1}{q_{j-1}} M_{j-2}^2 \geq \frac{1}{q_{j-1}} \frac{m_{j-3}^2}{q_{j-3}} \\ &\cdots \\ &\geq \frac{m_{j_0+1}^2}{q_{j-1} \cdots q_{j_0+1}} \text{ or } \frac{m_{j_0+2}^2}{q_{j-1} \cdots q_{j_0+2}}. \end{aligned}$$

On the other hand, it implies from (2.5) that $\lim_{j\to\infty} m_{j+1}^2 = \infty$. This contradicts the fact that $Q$ is a continuous function and proves our assertion. ∎

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