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MATHEMATICAL TREATMENT OF SMOLDERING COMBUSTION UNDER MICRO-GRAVITY

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Abstract. Various finger-like smoldering patterns are observed in experiments under micro-gravity. For theoretical understanding of such pattern phenomena, a model of reaction-diffusion system has been proposed. In this paper, we prove the existence and uniqueness of a solution for this reaction-diffusion system. We also consider a large-time behavior of solutions and show nonexistence results of traveling wave solutions.

Key words. Reaction-diffusion equations, numerical analysis, combustion.

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1. Introduction. It is shown in [4] that thin solid, for an example, paper, cellulose dialysis bags and polyethylene sheets, burning against oxidizing wind develops *finger-like patterns* or *fingering patterns*. The oxidizing gas is supplied in a uniform laminar flow, opposite to the directions of the front propagation and they control the flow velocity of oxygen, denoted by V_{O2}. When V_{O2} is decreased below a critical value, the smooth front develops a structure which marks the onset of instability. As V_{O2} is decreased further, the peaks are separated by cusp-like minima and a fingering pattern is formed. In addition, thin solid is stretched out straight onto the bottom plate and they also control the adjustable vertical gap, denoted by a parameter h, between top and bottom plates. Experimentally, fingering pattern occurs for small h, which implies that fingering pattern appears in the absence of natural convection. Similar phenomena have been observed in a micro-gravity experiment in space (see [2]).

Here we propose a phenomenological model described by the following reaction-diffusion system for the (dimensionless) temperature u, the density of paper v, the concentration of the mixed gas w.

$$(RD)\begin{cases} \dfrac{\partial u}{\partial t} = & Le\Delta u + \lambda\dfrac{\partial u}{\partial x} + \gamma k(u)vw - au^m, \quad (x,y)\in I\times\Omega, t>0,\\ \dfrac{\partial v}{\partial t} = & -k(u)vw, \quad (x,y)\in I\times\Omega, t>0,\\ \dfrac{\partial w}{\partial t} = & \Delta w + \lambda^{'}\dfrac{\partial w}{\partial x} - k(u)vw, \quad (x,y)\in I\times\Omega, t>0,\end{cases}$$

where the constants Le, called *Lewis number*, a and γ are positive parameters, λ and $\lambda^{'}$ are nonnegative parameters, $k(u)$ is a nonlinear term called *Arrhenius kinetics* and defined by $k(u) = exp(-1/u)$, $I \subset (-\infty,\infty)$ is a bounded interval $(0,l_x)$ or a whole line $(-\infty,\infty)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\Delta = \partial^2/\partial x^2 + \sum_{i=1}^n \partial^2/\partial y_i^2$ is

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Laplacian as usual. We suppose that if $I = (0, l_x)$, u, w satisfy

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(l_x, y, t) = 0, \quad \frac{\partial w}{\partial x}(0, y, t) = 0, \quad w(l_x, y, t) = w_r > 0$$

for any $y \in \Omega$ and $t > 0$, and if $I = (-\infty, \infty)$,

$$\lim_{|x| \to \infty} u(x, y, t) = 0, \quad \lim_{x \to \infty} w(x, y, t) = w_r, \quad \lim_{x \to -\infty} w(x, y, t) = w_l \geq 0$$

for any $y \in \Omega$ and $t > 0$. In both cases we also suppose that u, w satisfy

$$\frac{\partial u}{\partial \nu}(x, y, t) = \frac{\partial u}{\partial \nu}(x, y, t) = 0, \quad \frac{\partial w}{\partial \nu}(x, y, t) = \frac{\partial w}{\partial \nu}(x, y, t) = 0$$

for $x \in I$, $y \in \partial\Omega$ and $t > 0$, where ν is the unit exterior normal vector on $\partial\Omega$. We suppose that initial values u_0, v_0 and w_0 satisfy

$$u(x, y, 0) = u_0(x, y) \geq 0, \quad v(x, y, 0) = v_0(x, y) \geq 0, \quad w(x, y, 0) = w_0(x, y) \geq 0.$$

In numerical simulations, we take $\lambda = 0$ and $\lambda^{'}$ as a controlled parameter. If $\lambda^{'}$ is large, a smooth flame front is observed (see Figure 1.1 (a)). When $\lambda^{'}$ is decreased, the instability of a smooth flame front occurs. As $\lambda^{'}$ is decreased further, a fingering pattern is formed (see Figure 1.1 (b), (c)). Numerical simulations suggest that the model (RD) exhibits a qualitative agreement with the experimental results. This motives us to discuss analytically (RD) from pattern formation viewpoints. As the first step, we will show the existence and uniqueness of global solution of (RD) and to study the asymptotic behavior of the global solution.

(a) (b) (c)

large $\longleftarrow$ $\lambda^{'}$ $\longrightarrow$ small

FIG. 1.1. *various patterns in (RD)*

This paper is organized as follows; In Section 2, we show the global existence and uniqueness of a solution of (RD) (Theorems 2.1, 2.3). Furthermore we have the upper bound of a solution of (RD) (Lemma 2.2). In Section 3, we consider the asymptotic behavior of a global solution given in Section 2 (Theorem 3.1). In Section 4, we obtain the nonexistence results of a traveling wave solution (Lemmas 4.1, 4.3). ¿From experiments and simulations, we expect that there is a stable traveling wave solution if $\lambda^{'}$ is large. Then we would like to prove the existence of a traveling wave solution and in general, however, it is difficult. Hence it is necessary to obtain such conditions as there are no traveling wave solutions. We prove that there are no traveling wave solutions if a or λ is large. In addition, we also obtain the upper bound of the wave speed of a traveling wave solution in Lemma 4.3.

2. Existence and uniqueness of a global solution. In this section, we prove the existence and uniqueness of a global solution. We first prove the existence and uniqueness of a local solution. Then we replace w by z such as $w = z + \omega$, where $\omega = \omega(x)$ is a smooth positive function and satisfies $\omega(l_x) = w_r$ and $\omega^{'}(0) = 0$ if $I = (0, l_x)$, or $\omega \to w_r$ as $x \to \infty$ and $\omega \to w_l$ as $x \to \infty$ if $I = (-\infty, \infty)$. Then we consider the following system derived from (RD) with respect to (u, v, z);

$$\begin{cases} \dfrac{\partial u}{\partial t} = & Le\Delta u + \lambda \dfrac{\partial u}{\partial x} + \gamma k(u)v(z+\omega) - au^m, \qquad (x,y) \in I \times \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = & -k(u)v(z+\omega), \qquad (x,y) \in I \times \Omega, t > 0, \\ \dfrac{\partial z}{\partial t} = & \Delta z + \lambda^{'} \dfrac{\partial z}{\partial x} - k(u)v(z+\omega) + \omega^{''} + \lambda^{'}\omega^{'}, \quad (x,y) \in I \times \Omega, t > 0. \end{cases} \tag{2.1}$$

The initial values u_0, v_0 and z_0 are

$$\begin{gathered} u(x,y,0) = u_0(x,y) \geq 0, \quad v(x,y,0) = v_0(x,y) \geq 0, \\ z(x,y,0) = w_0(x,y) - \omega(x) \equiv z_0(x,y) \end{gathered} \tag{2.2}$$

for $x \in I$ and $y \in \Omega$. We suppose that u, z satisfy

$$\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(l_x,y,t) = 0, \quad \frac{\partial z}{\partial x}(0,y,t) = 0, \quad z(l_x,y,t) = 0 \tag{2.3}$$

for $y \in \Omega$ and $t > 0$ if $I = (0, l_x)$, and

$$\lim_{|x|\to\infty} u(x,y,t) = \lim_{|x|\to\infty} z(x,y,t) = 0 \tag{2.4}$$

for $y \in \Omega$ and $t > 0$ if $I = (-\infty, \infty)$. In both cases we suppose that u, z satisfy

$$\frac{\partial u}{\partial \nu}(x,y,t) = \frac{\partial u}{\partial \nu}(x,y,t) = 0, \quad \frac{\partial z}{\partial \nu}(x,y,t) = \frac{\partial z}{\partial \nu}(x,y,t) = 0, \tag{2.5}$$

for $x \in I$, $y \in \partial\Omega$ and $t > 0$.

We prove the existence and uniqueness of a local solution of the above system. In the proof, we shall use the standard theory of an analytic semigroup and prove the existence of the following integral equation;

$$\Phi(t) = T(t)\Phi_0 + \int_0^t T(t-s) f(\Phi(s)) ds, \tag{2.6}$$

where $\Phi = (u, v, z)^t$, $\Phi_0 = (u_0, v_0, z_0)^t$, $T(t)$ is a semigroup generated by a differential operator A defined by

$$A = \begin{pmatrix} Le\Delta + \lambda \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta + \lambda^{'} \dfrac{\partial}{\partial x} \end{pmatrix}$$

and

$$f(\Phi) = \begin{pmatrix} \gamma k(u)(\omega + z)v - au^m \\ -k(u)(\omega + z)v \\ -k(u)(\omega + z)v + \omega^{''} + \lambda^{'}\omega^{'} \end{pmatrix}.$$

We consider the integral equation (2.6) in the functional space X defined by

$$X = L^p(I \times \Omega) \times L^\infty(I \times \Omega) \times L^p(I \times \Omega)$$

for $p > 1$. And the domain of A, denoted by $D(A)$, is defined by

$$D(A) = W_N^{2,p}(I \times \Omega) \times L^\infty(I \times \Omega) \times W_{N,0}^{2,p}(I \times \Omega),$$

where if $I = (0, l_x)$, $W_N^{2,p}(I \times \Omega)$ is defined by

$$W_N^{2,p}(I \times \Omega) = \{u \in W^{2,p}(I \times \Omega) \mid \begin{array}{l} \dfrac{\partial u}{\partial \nu} = 0 \; for \; x \in I, \; y \in \partial\Omega, \\ \dfrac{\partial u}{\partial x} = 0 \; for \; x = 0, l_x, \; y \in \Omega\} \end{array}$$

and $W_{N,0}^{2,p}(I \times \Omega)$ is defined by

$$\begin{array}{l} W_{N,0}^{2,p}(I \times \Omega) = \{z \in W^{2,p}(I \times \Omega) \mid z = 0 \; for \; x = l_x, \; y \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0 \; for \; x \in I, \; y \in \partial\Omega, \quad \dfrac{\partial z}{\partial x} = 0 \; for \; x = 0, \; y \in \Omega\} \end{array}$$

and if $I = (-\infty, \infty)$, $W_N^{2,p}(I \times \Omega)$ is defined by

$$W_N^{2,p}(I \times \Omega) = \{u \in W^{2,p}(I \times \Omega) \mid \begin{array}{l} \dfrac{\partial u}{\partial \nu} = 0 \; for \; x \in I, \; y \in \partial\Omega, \\ \lim\limits_{|x| \to \infty} u(x, y) = 0 \; for \; y \in \Omega\} \end{array}$$

and $W_{N,0}^{2,p}(I \times \Omega)$ is equal to $W_N^{2,p}(I \times \Omega)$. $W^{2,p}(I \times \Omega)$ is a usual Sobolev space. We assume that $u_0 \in D(L_u^\alpha)$, $v_0 \in C^\kappa(I \times \Omega)$ and $z_0 \in D(L_z^\alpha)$ for $0 < \alpha < 1$ and $0 < \kappa < 1$, where $L_u = Le\Delta + \lambda\partial/\partial x$ and $L_z = \Delta + \lambda^{'}\partial/\partial x$. The functional spaces $D(L_u^\alpha)$ and $D(L_z^\alpha)$ are called *fractional spaces* (see Section 2.6 of [3]). And $C^\kappa(I \times \Omega)$ is the Hölder space with a Hölder exponent $0 < \kappa < 1$. Then we have the following theorem for existence of a local solution.

Theorem 2.1. *Assume that $p > n + 1$, $1/2 < \alpha < 1$, $0 < \kappa < 1$, and $\partial\Omega \in C^2$. In addition, suppose that the function ω has the second order Hölder continuous derivatives in $x \in I$ belonging to $L^p(I \times \Omega)$. Then, for any $(u_0, v_0, z_0) \in D(L_u^\alpha) \times C^\kappa(I \times \Omega) \times D(L_z^\alpha)$, there exist $T > 0$ and a unique local classical solution (u, v, z) of (2.1), (2.2), (2.3), and (2.5) if $I = (0, l_x)$, or (2.1), (2.2), (2.4), and (2.5) if $I = (-\infty, \infty)$ for $t < T$.*

We can show Theorem 2.1 by a standard argument. So we omit the details. In fact, the local solution obtained in Theorem 2.1 exists globally. To prove it, we need to obtain a priori estimate. We shall prove that u, v and w are bounded from above.

Lemma 2.2. *Let (u, v, z) be a solution given in Theorem 2.1 and set $w = z + \omega$. Then there exists a constant $R > 0$, depending on initial values u_0, v_0 and w_0, such that for any $(x, y) \in I \times \Omega$, $t > 0$,*

$$0 \le u \le R, \quad 0 \le v \le R, \quad 0 \le w \le R.$$

Proof. We first have

$$0 \le v \le \|v_0\|_{L^\infty(I \times \Omega)}, 0 \le w \le \max\{\|w_0\|_{L^\infty(I \times \Omega)}, w_r, w_l\}$$

because $\tilde{v} = \|v_0\|_{L^\infty(I\times\Omega)}$ and $\tilde{w} = \max\{\|w_0\|_{L^\infty(I\times\Omega)}, w_r, w_l\}$ are super-solutions of v and w respectively. From the first equation of (2.1), we have

$$\frac{\partial u}{\partial t} = Le\Delta u + \lambda\frac{\partial u}{\partial x} + \gamma k(u)vw - au^m \le Le\Delta u + \lambda\frac{\partial u}{\partial x} + \gamma\tilde{v}\tilde{w}k(u) - au^m.$$

Here we set $\tilde{u} = \max\{\|u_0\|_{L^\infty(I\times\Omega)}, \sup\{u > 0 \mid \gamma\tilde{v}\tilde{w}k(u) - au^m > 0\}\}$. Note that $\tilde{u} < \infty$ because of $k(u) < 1$. Then we readily see that $\tilde{u}$ is a super-solution of u, so that we obtain $0 \le u \le \tilde{u}$. Thus the proof is completed. □

¿From the above lemma, the following theorem holds.

THEOREM 2.3. *Let (u, v, z) be a solution given in Theorem 2.1. Then (u, v, z) exists globally.*

Proof. In order to prove this theorem, we shall obtain $\|u\|_\alpha \equiv \|u\|_{L^p(I\times\Omega)} + \|L_u^\alpha u\|_{L^p(I\times\Omega)}$ and $\|z\|_\alpha \equiv \|z\|_{L^p(I\times\Omega)} + \|L_z^\alpha z\|_{L^p(I\times\Omega)}$ exist for all $t > 0$. First of all, we obtain the estimate of $\|u\|_{L^p(I\times\Omega)}$. Here let $T_u(t)$ be an analytic semigroup generated by L_u. Then there exist some constants $C_1 > 0$ and $\beta \in \mathbb{R}$ such that $\|T_u(t)\| \le C_1 e^{\beta t}$. Using $T_u(t)$ and (2.6), we have

$$\|u\|_{L^p(I\times\Omega)} \le C_1 e^{\beta t}\|u_0\|_{L^p(I\times\Omega)} + C\int_0^t e^{\beta(t-s)}\|u\|_{L^p(I\times\Omega)}ds,$$

where $C > 0$ is a constant. Here note that $k(u)/u$ is bounded from above for $u > 0$. From Gronwall's inequality, it follows that

$$\|u\|_{L^p(I\times\Omega)} \le C_1 e^{(\beta+C)t}\|u_0\|_{L^p(I\times\Omega)}.$$

Next we estimate the norm $\|L_u^\alpha u\|_{L^p(I\times\Omega)}$. Since $\|L_u^\alpha T(t)\| \le C_1 e^{\beta t}/t^\alpha$ holds for $t > 0$ (see Theorem 6.13 in Section 2 of [3]), we obtain by using (2.6)

$$\begin{aligned}\|L_u^\alpha u\|_{L^p(I\times\Omega)} &\le C_1 e^{\beta t}\|L_u^\alpha u_0\|_{L^p(I\times\Omega)} + \int_0^t \frac{C}{(t-s)^\alpha}e^{\beta(t-s)}\|u\|_{L^p(I\times\Omega)}ds \\ &\le C_1 e^{\beta t}\|L_u^\alpha u_0\|_{L^p(I\times\Omega)} + \frac{Ct^{1-\alpha}}{1-\alpha}e^{(\beta+C)t}.\end{aligned}$$

Therefore $\|u\|_\alpha$ exists globally. From a similar argument, it can also be shown that $\|z\|_\alpha$ exists globally. □

3. Asymptotic behavior of u and v. In this section we consider the asymptotic behavior of classical solutions of (RD).

THEOREM 3.1. *Set $I = (0, l_x)$ and let (u, v, z) be a solution given in Theorem 2.1. For any $(x, y) \in I \times \Omega$, $\lim_{t\to\infty} u(x, y, t) = 0$. Moreover, there exists $v_\infty(x, y) \in L^\infty(I \times \Omega)$ such that $\lim_{t\to\infty} v(x, y, t) = v_\infty(x, y)$ and the function v_∞ has a positive value at any points $(x, y) \in I \times \Omega$ where $v_0(x, y) > 0$.*

It is easy to see that there exists $v_\infty(x, y)$ such that $v(x, y) \to v_\infty(x, y)$ as $t \to \infty$ because v decreases monotonically. Therefore the remainder is to obtain the asymptotic behavior of u and positiveness of v_∞.

We need lemmas to prove Theorem 3.1. As the first step of the proof of Theorem 3.1, we consider the reaction term $k(u)vw$ and prove that it approach 0 as $t \to \infty$. This implies that either u, v or w approach 0.

LEMMA 3.2. *Let (u, v, z) be a solution given in Theorem 2.1 and set $w = z + \omega$. Then, for any $(x, y) \in I \times \Omega$, it holds that $k(u)vw \to 0$ as $t \to \infty$.*

Proof. The statement of the lemma is equivalent to $v_t \to 0$, and we prove it. By using $v \to v_\infty$, for any $\epsilon > 0$ and $(x, y) \in I \times \Omega$, there exists $T > 0$ such that for $t > T$,

$$|v(x, y, t) - v_\infty(x, y)| < \epsilon^2.$$

Then we see that

$$\left| \frac{v(x, y, t + \epsilon) - v(x, y, t)}{\epsilon} \right| \leq 2\epsilon,$$

and then

$$|v_t(x, y, t + \theta\epsilon)| \leq 2\epsilon$$

for some $\theta \in (0, 1)$. Thus it follows that

$$\limsup_{t\to\infty} |v_t(x, y, t)| = \limsup_{t\to\infty} |v_t(x, y, t + \theta\epsilon)| \leq 2\epsilon.$$

Since ϵ is any small parameter, we have $\lim_{t\to\infty} v_t(x, y, t) = 0$, which completes the proof. □

Here we note that we can also show the similar result to Lemma 3.2 in the case of $I = (-\infty, \infty)$.

The previous lemma implies that u, v or w approach 0. Now we define a constant M_k for integers $k \geq 1$ by $M_k \equiv \sup_{u>0} k(u)/u^k$. Using this constant, we have $k(u) \leq M_m u^m$.

LEMMA 3.3. *Set $I = (0, l_x)$ and let (u, v, z) be a solution given in Theorem 2.1. Then it holds that $u \to 0$ as $t \to \infty$ at any $(x, y) \in I \times \Omega$.*

Proof. Set $w = z + \omega$. Since $k(u)vw \to 0$ from Lemma 3.2, there exists $T > 0$ such that $\gamma k(u)vw \leq au^m/2$ for any $t > T$ and $(x, y) \in I \times \Omega$. Then u satisfies

$$u_t \leq \Delta u + \lambda \frac{\partial u}{\partial x} - \frac{a}{2} u^m$$

for any $t > T$ and $(x, y) \in I \times \Omega$. Now we use a constant R given in Lemma 2.2 and define $q = q(t)$ by a solution of

$$q' = -\frac{a}{2} q^m, \quad q(T) = R.$$

The function q is explicitly written such as

$$q(t) = \begin{cases} \dfrac{R}{\{a(m-1)R^{m-1}(t-T)/2 + 1\}^{1/(m-1)}}, & m > 1, \\ R\, exp(-a(t-T)/2), & m = 1. \end{cases}$$

Using $u \leq R$ and applying the comparison principle to u and q, we have $u \leq q$ for $t > T$. Since $q \to 0$ as $t \to \infty$, u also approaches 0, which completes the proof. □

¿From the previous lemma, u does approach 0 as $t \to \infty$. Now we are in position to prove that $v_\infty > 0$.

Proof. Let $q = q(t)$ be a function given in the proof of Lemma 3.3. As stated previously, we have $0 \leq u \leq q$ for any $t > T, (x, y) \in I \times \Omega$. Then it follows from the second equation of (RD) that

$$v_t \geq -k(u)wv \geq -M_m R q^m v,$$

where R is a constant given in Lemma 2.2. By using this inequality, we obtain

$$\begin{aligned} v(x,y,t) &\geq v(x,y,T)exp(-M_mR\int_T^t q^m ds) \\ &= v(x,y,T)exp(-\frac{2M_mR}{a}(q(T)-q(t))) \geq v(x,y,T)exp(-\frac{2M_mR}{a}q(T)). \end{aligned} \tag{3.1}$$

Here we have an estimate of $v(x,y,T)$ such as

$$v(x,y,T) = v_0(x,y)exp(-\int_0^T k(u)wds) \geq v_0(x,y)exp(-RT), \tag{3.2}$$

because of $w \leq R$ and $k(u) < 1$. Therefore it follows from (3.1) and (3.2) that

$$v(x,y,t) \geq v_0(x,y)exp(-RT - \frac{2M_mR}{a}q(T)),$$

which implies that $v_\infty(x,y) > 0$ if $v_0(x,y) > 0$. □

4. Nonexistence of a traveling wave solution. ¿From the results of experiments and simulations, we expect that there exists a traveling wave solution independent of y-direction and moving opposite to the flow of the oxidizing gas, which is a positive solution of (RD) and can be written such as

$$u(x,t) = U(x-ct), \quad v(x,t) = V(x-ct), \quad w(x,t) = W(x-ct),$$

where $c > 0$ is a wave speed. Here we set $z = x - ct$. Then our equations satisfied by (U,V,W) have the forms

$$\begin{aligned} -cU' &= LeU'' + \lambda U' + \gamma k(U)VW - aU^m, \\ -cV' &= -k(U)VW, \\ -cW' &= W'' + \lambda' W' - k(U)VW, \end{aligned} \tag{4.1}$$

where the prime $'$ is d/dz, and boundary conditions are

$$U(\pm\infty) = 0, \quad V(+\infty) = V_r > 0, \quad W(+\infty) = W_r > 0, \quad W(-\infty) = W_l(< W_r).$$

In this paper, we consider the conditions of the parameters as there does not exist a traveling wave solution, although we would like to prove the existence of a traveling wave solution (U,V,W,c). Recalling the constant M_m, defined before Lemma 3.3.

LEMMA 4.1. *If $a \geq \gamma M_m V_r W_r$, there does not exist a traveling wave solution.*

Proof. We first prove that $V \leq V_r$ and $W \leq W_r$ whenever a traveling wave solution (U,V,W,c) exists. From the second equation of (4.1), V increases monotonically and satisfies $V \leq V_r$. Next we suppose that $W > W_r$ holds. Then W must have the maximum at some point. Hence it follows from the third equation of (4.1) and $W' = 0$ that

$$W'' = k(U)VW > 0$$

at the point, which is a contradiction. Therefore we have $W \leq W_r$.

Next assume that there exists a traveling wave solution (U,V,W,c) for $a \geq \gamma M_m V_r W_r$. From the boundary condition, the function U must have the maximum at some point. Then we have from the first equation of (4.1) and $U' = 0$,

$$LeU'' = aU^m - \gamma k(U)VW \geq U^m(a - \gamma M_m V_r W_r) \geq 0,$$

which is a contradiction. □

We showed in the previous lemma that no traveling wave solutions exist if $a \geq \gamma M_m V_r W_r$. In addition, if $a > \gamma M_m V_r W_r$, we have the asymptotic behavior of the function u of a solution of (RD) as $t \to \infty$.

LEMMA 4.2. *Let (u, v, w) be a solution given in Theorem 2.1 with the initial conditions (u_0, v_0, w_0) satisfying $v_0 \leq V_r$ and $w_0 \leq W_r$. If $a > \gamma M_m V_r W_r$, $\|u\|_{L^\infty(I\times\Omega)} \to 0$ as $t \to \infty$.*

This result implies that flame front does not keep spreading with high temperature if heat radiation is too strong.

Proof. From the assumption, we have from the first equation of (RD)

$$u_t \leq Le\Delta u + \lambda\frac{\partial u}{\partial x} - (a - \gamma M_m V_r W_r)u^m.$$

Let $q = q(t)$ be a solution of

$$q' = -(a - \gamma M_m V_r W_r)q^m, \quad q(0) = R,$$

where R is a constant given in Lemma 2.2. Then it holds that $u \leq q$ and $q \to 0$ as $t \to \infty$. This completes the proof. □

Next we show that there does not exist traveling wave solutions if λ is large.

LEMMA 4.3. *Set $m = 1$. Then there exists a constant c^* independent of Le, λ such that (4.1) possesses no traveling wave solutions with the wave speed c satisfying $c > \sqrt{Le}c^* - \lambda$.*

¿From this lemma, we can state that the wave speed of traveling wave solutions must satisfy $c \leq \sqrt{Le}c^* - \lambda$ if exists. This is the upper bound of the wave speed.

Proof. From Lemma 4.1, we can assume $a < \gamma M_1 V_r W_r$ without loss of generality. Here we set $f(U) = \gamma k(U)V_r W_r - aU$. Then $f(U)$ has zeros at $U = 0, U_1, U_2$ for $0 < U_1 < U_2$. Moreover it holds that $f(U) < 0$ for $0 < U < U_1$ and $f(U) > 0$ for $U_1 < U < U_2$, which implies that $f(U)$ is a nonlinear term of a bistable type. Therefore there exists a traveling wave solution $Q = Q(\tilde{z})$ with the wave speed c^* independent of Le, λ such that

$$-\sqrt{Le}c^*Q' = LeQ'' + f(Q), \quad Q(-\infty) = U_2, \quad Q(+\infty) = 0,$$

where $\tilde{z} = x - (\sqrt{Le}c^* - \lambda)t$ and the prime $' = d/d\tilde{z}$ (see Theorem 4.2 of [1]). Setting $q(x,t) = Q(x - (\sqrt{Le}c^* - \lambda)t)$, we know that q satisfies

$$\frac{\partial q}{\partial t} = Le\frac{\partial^2 q}{\partial x^2} + \lambda\frac{\partial q}{\partial x} + f(q).$$

Now we assume that there exists a traveling wave solution (U, V, W) with the wave speed $c > \sqrt{Le}c^* - \lambda$. We have obtained $V \leq V_r$ and $W \leq W_r$ in Lemma 4.1. Then $u(x,t) = U(x - ct)$ satisfies

$$\frac{\partial u}{\partial t} \leq Le\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial u}{\partial x} + f(u)$$

and $u < U_2$ for $-\infty < x < \infty$. Then we can show that $q(x,t)$ is a super solution of $u(x,t)$. To prove it, it should be shown that u decays faster as $x \to \infty$ than q. Indeed, this is true because u decays as $x \to \infty$ at the rate $exp(\mu_1 x)$, where $\mu_1 = \{-(\lambda + c) - \sqrt{(\lambda + c)^2 + 4aLe}\}/2Le$, and q does at the rate $exp(\mu_2 x)$, where

$\mu_2 = \{-\sqrt{Le}c^* - \sqrt{Le(c^*)^2 + 4aLe}\}/2Le$. Hence for some $h \in \mathbb{R}$, we have $u(x,t) < q(x-h,t)$ for any $-\infty < x < \infty$ and $t > 0$, that is, $U(x-ct) < Q(x-h-(\sqrt{Le}c^*-\lambda)t)$. However, since $c > \sqrt{Le}c^* - \lambda$, U must reach Q at some point in a finite time, which is a contradict. □

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