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# COMPUTER ANALYSIS OF FRACTAL SETS

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**Abstract.** This article deals with the numerical computation of the Box-counting dimension of fractal sets. The improvement of this method is presented and successful results are shown.

**Key words.** Fractal sets, box-counting dimension, numerical computation.

**AMS subject classifications.** 35L65, 76M12, 80A20

**1. Introduction.** Fractal geometry is one of the relatively new mathematical disciplines. Unlike classical geometry, it can be used for a better approximation of objects in the real nature. Fractal geometry deals with highly irregular objects as well as with smooth objects studied by classical geometry (see [6], [7], and [5]).

Because of its adequate representation of many natural phenomena, fractal geometry is often used in practical applications, namely in biology, medical science, and mechanical engineering. Fractal geometry can simulate growth of various living things starting from simple plants and ending with tumors where it can help curing patients with cancer. In mechanical engineering, fractal geometry is for example used for measuring complexity of cracks in material (see [5]).

Complexity, in the sense of fractal geometry, is measured by the fractal dimension. Unlike the classical dimension, the fractal dimension can be any non-negative real number and can be estimated in several ways. This article will present the two most important types of dimensions, i.e. the Hausdorff dimension and Box-counting dimension. The improvement of the standard numerical computation of Box-counting will be introduced and the successful result will be shown (see [1]).

**2. Hausdorff measure and Hausdorff dimension.** The Hausdorff measure is, in the field of fractal geometry, one of the most useful generalizations of measure. Using this measure, we can define the Hausdorff dimension, a dimension that can be used for almost any subset of $\mathbb{R}^n$. Its main disadvantage is that it is not easy to determine, and it is difficult to be numerically approximated. The Hausdorff measure (see [1]) is defined as follows:

Let

$$\bar{\mathcal{H}}^s_\epsilon(F) = \inf \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^s, \tag{2.1}$$

where the infimum is over all possible $\epsilon$-covers $\mathcal{A}$ of the set $F$. Then the outer Hausdorff measure is defined as follows:

$$\bar{\mathcal{H}}^s(F) = \lim_{\epsilon \to \infty} \bar{\mathcal{H}}^s_\epsilon(F). \tag{2.2}$$

This measure is 0 or $\infty$ for almost every value of $s$. The value of $s$, where the measure changes from $\infty$ to 0, is called *the Hausdorff dimension* [1]. Therefore:

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \bar{\mathcal{H}}^s(F) = 0\} = \sup\{s \geq 0 : \bar{\mathcal{H}}^s(F) = \infty\}. \tag{2.3}$$

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The Hausdorff dimension measures complexity of a set in such a way that, for example, irregular curves in a plane have a dimension higher than 1 but lower than 2 because they are not as complex as a smooth surface. Similarly, irregular surfaces have dimensions between 2 and 3.

**3. Box-counting dimension.** The Box-counting dimension is similar to the Hausdorff dimension, however, its definition considers sets with a fixed diameter. As a consequence, the numerical approximation is easier. The Box-counting dimension (see [1]) is defined as follows:

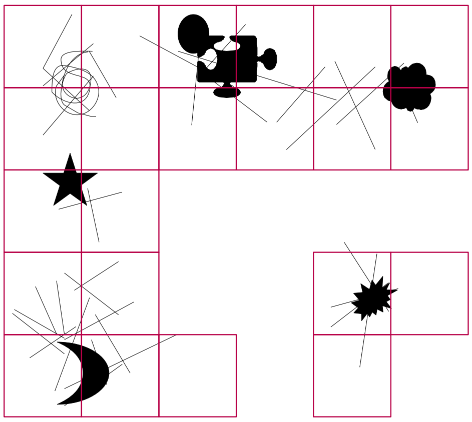

Fig. 3.1: Covering used for numerical estimation of the Box-counting dimension. $N_\delta(F)$ represents the number of squares containing some points of the set $F$.

Definition 3.1. *Let $F \subset \mathbb{R}^n$ and $N_\delta(F)$ be the smallest number of sets with diameter $\delta$ which covers the set $F$. We define the Box-counting dimension as:*

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log_{\delta_k}}. \tag{3.1}$$

It has been proved (e.g. in [1]) that the Box-counting dimension equals the Hausdorff dimension for a large class of sets. Furhermore, we can use various types of sets for covering (see [1]). For example, we can use circles of diameter $\delta$ only, squares of side $\delta$, or squares in the mesh with a step of $\delta$. The last option is used frequently in numerical approximations of the Box-counting dimension.

**4. Numerical approximation of the Box-counting dimension.** The Box-counting dimension can be approximated as follows (see [1], [2], [3]). The set is covered with a $\delta$-mesh for various values of $\delta$. For each $\delta$, the number of mesh squares containing some part of the set $F$ is counted. These values can be presented into a log-log graph, where the $y$-axis corresponds to $\log N_\delta(F)$ and the $x$-axis corresponds to $-log\delta$. The regressively obtained value of the slope of this graph approximates the Box-counting dimension. Since this algorithm is not entirely accurate, we suggest its modification.

**5. Improvement of the Box-counting dimension.** One of the main problems coming with the Box-counting dimension is that the values in the log-log graph

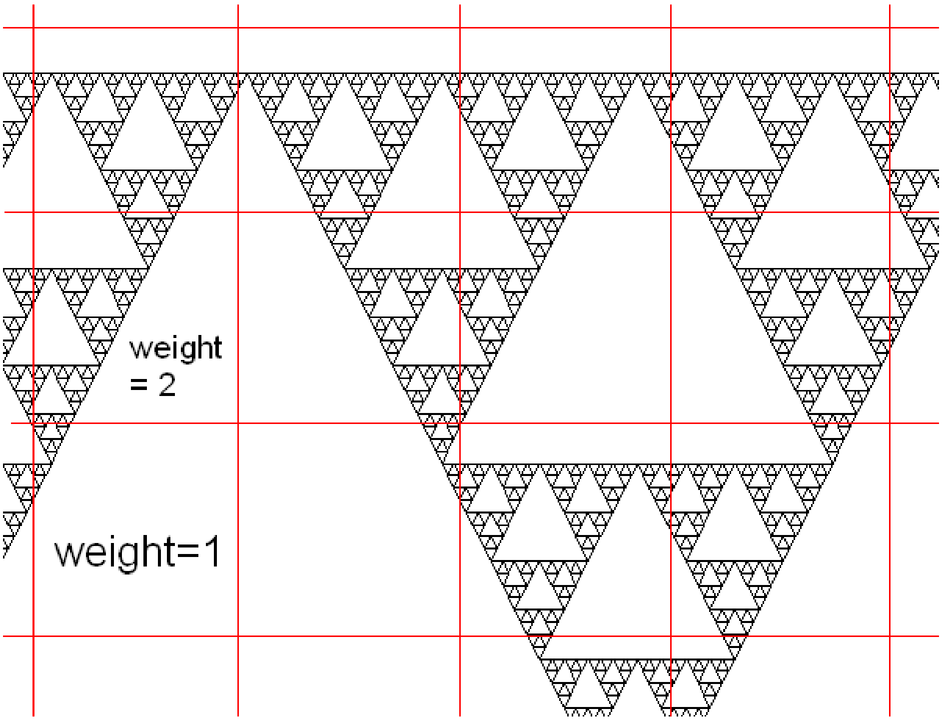


Fig. 4.1: Assigning weights. Squares with a small number of points of the set $F$ get lower weight than squares with many points.

do not lie on a line. Therefore, the linear regression has to be used to estimate the slope. Additionally, a reliable result requires many measurements. We suggest to improve the approximation of the dimension by assigning weight to every square in a mesh. The details are described below.

First, the number of points of the set $F$ for every square in a mesh is counted. We find the maximal value ($max$) and then we assign weights according to how many points each square contains. The following two ways of weight assignment yield reasonable results.

Method I.: Each square with zero points has a weight equal to 0. Squares with the number of points in the interval $[1, max/2]$ have a weight equal to 1, and all other squares have the weight equal to 2.

Method II.: Each square with zero points has a weight equal to 0. Squares with the number of points in the interval $[1, max/3]$ have a weight equal to 1, squares with the number of points in the interval $[max/2, max/3]$ have a weight equal to 2, and all other squares have a weight equal to 3.

Consequently, we sum the weights and perform the linear regression to obtain the estimate of the dimension. By assigning the weights, smoothing of data has been achieved and accuracy has been improved. The weights also contributed to speeding up the convergence.

**6. Results.** The following table shows the results of our computation. The table presents couples of characteristics - dimension estimate and the computation error.

We can see that the method 2 gives good results. The algorithm is slightly slower than the algorithm for the standard Box-counting dimension, but still the time of computation does not exceed few seconds on a standard PC.

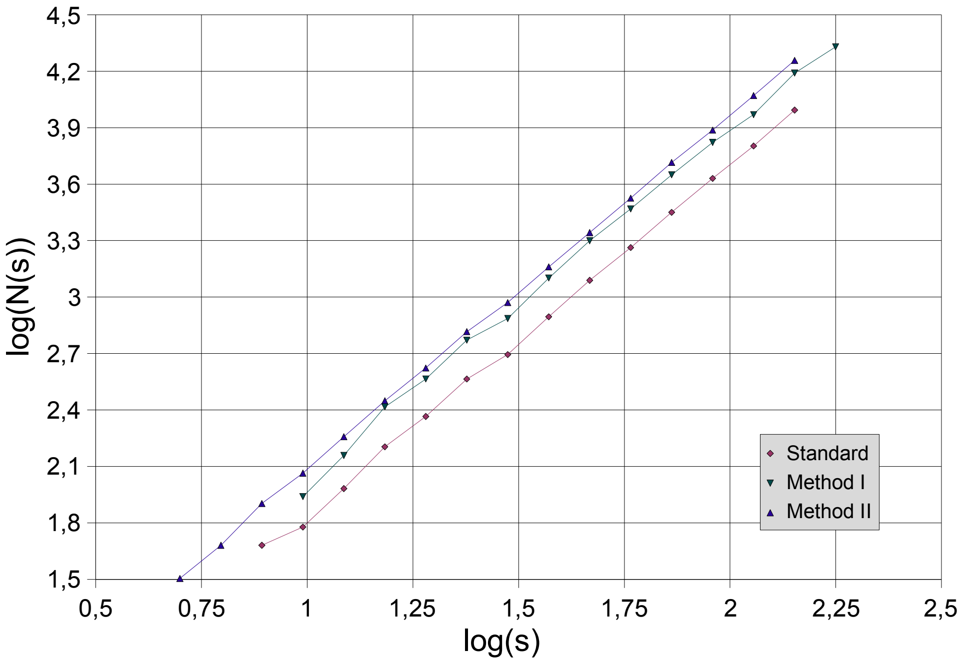


Fig. 6.1: Example of log-log graph for the Sierpiński carpet. We can see that the graph for the Method II. is very close to a line.

| Set | Normal | Method I. | Method II. | real |
|---|---|---|---|---|
| line | 0.9636; 3.63% | 0.9874; 1.25% | 0.9934; 0.66% | 1 |
| square | 1.9746; 1.27% | 1.9717; 1.41% | 1.9966; 0.17% | 2 |
| Circle | 1.0411; 4.12% | 0.9856; 1.43% | 0.9951; 0.48% | 1 |
| Sierp. triangle | 1.6093; 1.53% | 1.5985; 0.85% | 1.6021; 1.08% | $\log 3/\log 2$ |
| Sierp. carpet | 1.8631; 1.57% | 1.8718; 1.11% | 1.8839; 0.47% | $\log 8/\log 3$ |
| Koch curve | 1.2866; 1.96% | 1.2771; 1.21 % | 1.2756; 1.09% | $\log 4/\log 3$ |
| Cantor discont. | 1.0085; 0.85% | 0.9958; 0.42% | 0.9996; 0.04% | 1 |

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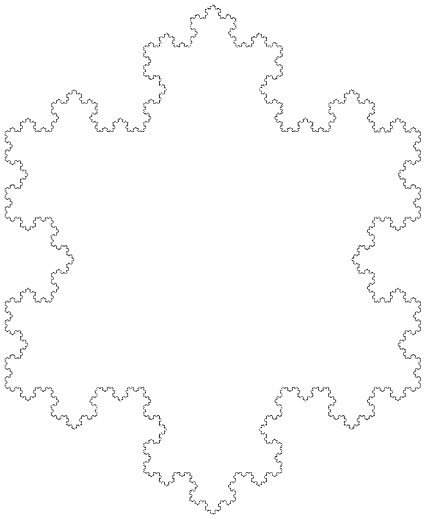

Fig. 6.2: Koch Snowflake. The Hausdorff dimension is equal to $\log 4/\log 3$. Numerical estimation (Method II.) gives the value of 1.2756 with error of 1.09%.

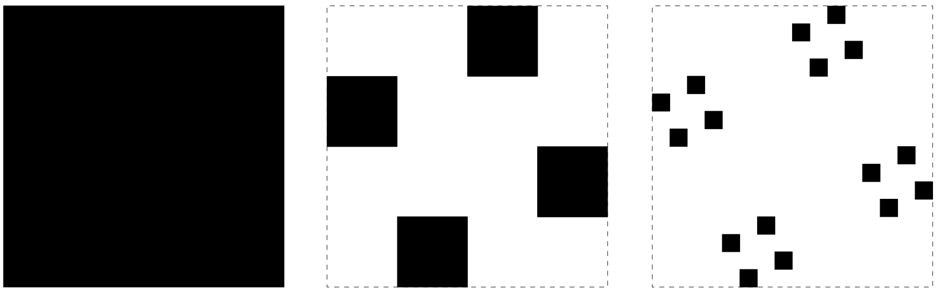

Fig. 6.3: Cantor Discontinuum. The Hausdorff dimension is equal to 1. Numerical estimation (Method II.) gives the value of 0.9996 with error of 0.04%.

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