Geometry of hypersurfaces and moving hypersurfaces in $\mathbb{R}^m$

— for the study of moving boundary problems —

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Plan of the lectures

I. Geometry of hypersurfaces

- Differential operators on a hypersurface Γ
  (gradient, divergence, Laplace-Beltrami operator on Γ)
- Weingarten map and principal curvatures of Γ
- Gauss-Green formula on Γ

II. Applications

- Signed distance function and level set functions
- Hanzawa transform
- General curvilinear coordinates
- Generalized Frenet's formula

III. Moving hypersurfaces

- Normal velocity of $\Gamma(t)$
- Normal time derivative
- Signed distance function for moving hypersurfaces

IV. Variational formulas

- Transport identities
- Variation of geometic quantities
- Gauss-Bonnet formula
- Moving boundary problems
  (mean curvature flow, Gauss curvature flow, Hele-Shaw flow, Wilmore flow)

1 Geometry of hypersurfaces

1.1 Curvature of plane curves

We start from plane curves. Let Γ be a C^2-class curve in $\mathbb{R}^2$ parametrized by a length parameter $s \in \mathcal{I}$, where $\mathcal{I}$ is an interval:

$$\Gamma = \{\boldsymbol{\gamma}(s) = (\gamma_1(s), \gamma_2(s))^{\mathrm{T}} \in \mathbb{R}^2;\ s \in \mathcal{I}\}, \quad \boldsymbol{\gamma} \in C^2(\mathcal{I}, \mathbb{R}^2), \quad |\boldsymbol{\gamma}'(s)| = 1.$$

The tangential and normal unit vectors are given by

$$\boldsymbol{\tau}(s) := \boldsymbol{\gamma}'(s), \qquad \boldsymbol{\nu}(s) := (-\gamma_2'(s), \gamma_1'(s))^{\mathrm{T}}.$$

Exercise 1.1.1. *Prove that* $\boldsymbol{\tau}'(s) \parallel \boldsymbol{\nu}(s)$ *and* $\boldsymbol{\nu}'(s) \parallel \boldsymbol{\tau}(s)$.

The signed curvature $\kappa(s)$ is defined by the formulas

$$\boldsymbol{\tau}'(s) = \kappa(s)\boldsymbol{\nu}(s), \qquad \boldsymbol{\nu}'(s) = -\kappa(s)\boldsymbol{\tau}(s).$$

These are known as the Frenet formula for plane curves. Please be careful of the sign convention of $\boldsymbol{\nu}$ and κ.

Example 1.1.2.

$$\Gamma = \{\boldsymbol{x} \in \mathbb{R}^2;\ |\boldsymbol{x}| = r\}, \qquad \boldsymbol{\gamma}(s) = \left(r\cos\left(\frac{s}{r}\right),\ r\sin\left(\frac{s}{r}\right)\right)^{\mathrm{T}} \quad (0 \le s < 2\pi r),$$

$$\boldsymbol{\tau}(s) = \left(-\sin\left(\frac{s}{r}\right),\ \cos\left(\frac{s}{r}\right)\right)^{\mathrm{T}}, \qquad \boldsymbol{\nu}(s) = -\left(\cos\left(\frac{s}{r}\right),\ \sin\left(\frac{s}{r}\right)\right)^{\mathrm{T}}$$

$$\boldsymbol{\tau}'(s) = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right),\ -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right)^{\mathrm{T}} = \frac{1}{r}\boldsymbol{\nu}(s), \qquad \kappa(s) = \frac{1}{r}.$$

Exercise 1.1.3. *For* Γ *given by a graph* $\eta = u(\xi)$ $(\xi \in \mathcal{I}_1 \subset \mathbb{R})$ *in* ξ-η *plane with* $u \in C^2(\mathcal{I}_1)$ *and* $\frac{ds}{d\xi} > 0$, *prove that*

$$\kappa = \frac{u''(\xi)}{(1 + |u'(\xi)|^2)^{\frac{3}{2}}}.$$

Exercise 1.1.4. *For* $\Gamma = \{\boldsymbol{\varphi}(\xi) = (\varphi_1(\xi), \varphi_2(\xi))^{\mathrm{T}} \in \mathbb{R}^2;\ \xi \in \mathcal{I}_2 \subset \mathbb{R}\}$ *with* $\varphi \in C^2(\mathcal{I}_2)$ *and* $\frac{ds}{d\xi} > 0$, *prove that*

$$\kappa = \frac{\varphi_1'(\xi)\varphi_2''(\xi) - \varphi_1''(\xi)\varphi_2'(\xi)}{|\boldsymbol{\varphi}'(\xi)|^3}.$$

1.2 Principal curvatures of hypersurfaces

Let $m \in \mathbb{N}$ and $m \ge 2$. We consider an oriented C^2-class hypersurface Γ in $\mathbb{R}^m$, i.e., there is a vector field $\boldsymbol{\nu} \in C^1(\Gamma, \mathbb{R}^m)$ such that

$$|\boldsymbol{\nu}(\boldsymbol{x})| = 1, \qquad \boldsymbol{\nu}(\boldsymbol{x}) \in \mathrm{T}_{\boldsymbol{x}}(\Gamma)^{\perp} \qquad (\boldsymbol{x} \in \Gamma),$$

where

$$\mathrm{T}_{\boldsymbol{x}}(\Gamma) := \{\boldsymbol{y} \in \mathbb{R}^m;\ \boldsymbol{y} \cdot \boldsymbol{\nu}(\boldsymbol{x}) = 0\} \subset \mathbb{R}^m,$$

denotes the tangent space of Γ at $\boldsymbol{x}$ and $\boldsymbol{\nu}$ represents the unit normal vector field on Γ.

For a point on Γ, without loss of generality we assume that it is the origin $\mathbf{0} \in \Gamma \subset \mathbb{R}^m$, we choose a suitable orthogonal coordinate $(\xi_1, \xi_2, \cdots, \xi_{m-1}, \eta)^{\mathrm{T}}$ of $\mathbb{R}^m$ such that the tangent space and the unit normal vector at $\mathbf{0} \in \Gamma$ are given by

$$\mathrm{T}_{\mathbf{0}}(\Gamma) = \left\{ \begin{pmatrix} \boldsymbol{\xi} \\ 0 \end{pmatrix} \in \mathbb{R}^m;\ \boldsymbol{\xi} = (\xi_1, \cdots, \xi_{m-1})^{\mathrm{T}} \in \mathbb{R}^{m-1} \right\}, \qquad \boldsymbol{\nu}(\mathbf{0}) = (0, \cdot, 0, 1)^{\mathrm{T}} \in \mathbb{R}^m.$$

In a neighborhood of the origin, Γ is locally expressed by a graph $\eta = u(\boldsymbol{\xi})$ ($\boldsymbol{\xi} \in \mathcal{O}' \subset \mathbb{R}^{m-1}$), where

$$u \in C^2(\mathcal{O}'), \quad \nabla_\xi u(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^{m-1}.$$

We denote by $\nabla_\xi^2 u$ the Hessian matrix of $u(\boldsymbol{\xi})$

$$\nabla_\xi^2 u(\boldsymbol{\xi}) = \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(\boldsymbol{\xi}) \right)_{i,j=1,\cdots,m-1} \in \mathbb{R}^{(m-1)\times(m-1)}_{\mathrm{sym}}.$$

Definition 1.2.1 (principal curvatures, principal directions). *Let the eigenvalues and the eigenvectors of the symmetric matrix $\nabla_\xi^2 u(\mathbf{0}) \in \mathbb{R}^{(m-1)\times(m-1)}_{sym}$ be denoted by $(\kappa_i, \boldsymbol{e}_i') \in \mathbb{R} \times \mathbb{R}^{m-1}$ $(i = 1, \cdots, m-1)$, where $\boldsymbol{e}_i' = (e_{i1}, \cdots, e_{i(m-1)})^{\mathrm{T}}$, with*

$$\nabla_\xi^2 u(\mathbf{0}) \boldsymbol{e}_i' = \kappa_i \boldsymbol{e}_i', \qquad \boldsymbol{e}_i' \cdot \boldsymbol{e}_j' = \delta_{ij}.$$

Then κ_i $(i = 1, \cdots, m-1)$ are called the principal curvatures of Γ at $(\boldsymbol{\xi}, \eta) = \mathbf{0} \in \Gamma$, and $\boldsymbol{e}_i \in \mathrm{T}_{\mathbf{0}}(\Gamma) \subset \mathbb{R}^m$ defined by $\boldsymbol{e}_i := (e_{i1}, \cdots, e_{i(m-1)}, 0)^{\mathrm{T}}$ is called the principal direction with respect to κ_i.

We define $\kappa := \sum_{i=1}^{m-1} \kappa_i$, and we call κ the mean curvature in stead of $\kappa/(m-1)$ in this lecture. We also define the Gaussian curvature by $\kappa_g := \Pi_{i=1}^{m-1} \kappa_i$.

Proposition 1.2.2. *For $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_{m-1}, 0)^{\mathrm{T}} \in \mathrm{T}_{\mathbf{0}}(\Gamma)$ with $|\boldsymbol{\tau}| = 1$, we define a plane curve in $\boldsymbol{\tau}$-η plane with a coordinate $(\sigma, \eta)^{\mathrm{T}} \in \mathbb{R}^2$ by*

$$\Gamma_\tau := \left\{ (\sigma, u(\sigma \boldsymbol{\tau}'))^{\mathrm{T}} \in \mathbb{R}^2;\ \sigma \in \mathcal{I} \right\},$$

where $\boldsymbol{\tau}' = (\tau_1, \cdots, \tau_{m-1})^{\mathrm{T}} \in \mathbb{R}^{m-1}$. Then the curvature of Γ_τ at $(\sigma, \eta) = (0, 0)$ is given by

$$\kappa_\tau = (\boldsymbol{\tau}')^{\mathrm{T}} \left(\nabla_\xi^2 u(\mathbf{0}) \right) \boldsymbol{\tau}' = \sum_{i=1}^{m-1} \kappa_i |\boldsymbol{\tau} \cdot \boldsymbol{e}_i|^2.$$

In particular, $\kappa_{e_i} = \kappa_i$ holds.

The following lemma will be used in proving Theorem 1.3.6.

Lemma 1.2.3. *Under the condition of Proposition 1.2.2, we define*

$$\boldsymbol{x}(\sigma) := \sigma \boldsymbol{\tau} + u(\sigma \boldsymbol{\tau}') \boldsymbol{\nu}(\mathbf{0}) \in \Gamma.$$

Then we have

$$\left. \frac{d}{d\sigma} \boldsymbol{\nu}(\boldsymbol{x}(\sigma)) \right|_{\sigma=0} = - \begin{pmatrix} \nabla_\xi^2 u(\mathbf{0})\, \boldsymbol{\tau}' \\ 0 \end{pmatrix}.$$

1.3 Differential operators on Γ

We suppose that $\mathcal{O}$ is an open set of $\mathbb{R}^m$ such that $\Gamma \subset \mathcal{O} \subset \mathbb{R}^m$.

Lemma 1.3.1. *If $f \in C^1(\mathcal{O})$ and $f|_\Gamma = 0$, then $\nabla f(\boldsymbol{x}) \in \mathrm{T}_{\boldsymbol{x}}(\Gamma)^\perp$.*

Proof. We assume that Γ is locally parametrized as $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}) \in \Gamma \in \mathbb{R}^m$ by $\boldsymbol{\xi} \in \mathbb{R}^{m-1}$. Since $\mathrm{T}_{\boldsymbol{x}}(\Gamma) = \langle \frac{\partial \boldsymbol{\varphi}}{\partial \xi_1}(\boldsymbol{\xi}), \cdots, \frac{\partial \boldsymbol{\varphi}}{\partial \xi_{m-1}}(\boldsymbol{\xi}) \rangle$ for $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi})$, the assertion follows from

$$0 = \frac{\partial}{\partial \xi_i} f(\boldsymbol{\varphi}(\boldsymbol{\xi})) = (\nabla f(\boldsymbol{x}))^{\mathrm{T}} \ \frac{\partial \boldsymbol{\varphi}}{\partial \xi_i}(\boldsymbol{\xi}) \qquad (i = 1, \cdots, m-1).$$

□

Definition 1.3.2 (Gradient on Γ). *For $f \in C^1(\Gamma)$, we define*

$$\nabla_\Gamma f(\boldsymbol{x}) := \Pi_{\boldsymbol{x}} \, \nabla \tilde{f}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma),$$

where $\Pi_{\boldsymbol{x}} := (I - \boldsymbol{\nu}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})^{\mathrm{T}})$ is the orthogonal projection from $\mathbb{R}^m$ to $\mathrm{T}_{\boldsymbol{x}}(\Gamma)$, and $\tilde{f} \in C^1(\mathcal{O})$ is arbitarary C^1-extension of f to an open neighborhood of Γ with $\tilde{f}|_\Gamma = f$. From Lemma 1.3.1, $\nabla_\Gamma f$ does not depend on the choice of $\tilde{f}$.

Definition 1.3.3 (Divergence on Γ). *For $\boldsymbol{h} \in C^1(\Gamma, \mathbb{R}^m)$, we define* $\mathrm{div}_\Gamma \boldsymbol{h} := \mathrm{tr} \nabla_\Gamma \boldsymbol{h}^{\mathrm{T}}$.

Proposition 1.3.4. *For $\boldsymbol{h} \in C^1(\mathcal{O}, \mathbb{R}^m)$, we have* $\mathrm{div}_\Gamma \boldsymbol{h} = \mathrm{div} \boldsymbol{h} - \boldsymbol{\nu}^{\mathrm{T}}(\nabla^{\mathrm{T}} \boldsymbol{h})\boldsymbol{\nu}$ *on Γ .*

Definition 1.3.5 (Laplace-Beltrami operator). *For $f \in C^2(\Gamma)$, we define $\Delta_\Gamma f := \mathrm{div}_\Gamma \nabla_\Gamma f$.*

Theorem 1.3.6. *We define $W \in C^0(\Gamma, \mathbb{R}^{m \times m})$ by*

$$W(\boldsymbol{x}) := -\nabla_\Gamma^{\mathrm{T}} \boldsymbol{\nu}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Gamma.$$

Then, $W(\boldsymbol{x})$ is symmetric and

$$\begin{cases} W(\boldsymbol{x})\boldsymbol{e}_i = \kappa_i \boldsymbol{e}_i & (i = 1, \cdots, m-1) \\ W(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{0} \end{cases}$$

holds, where κ_i and $\boldsymbol{e}_i$ are the pricipal curvatures and the corresponding principal directions at $\boldsymbol{x} \in \Gamma$ with $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$.

Proof. We fix $\boldsymbol{x} \in \Gamma$. The equality $W(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = 0$ is clear from the definition of W. For each principal direction $\boldsymbol{e}_i$, there exists $\boldsymbol{y} \in C^2((-a, a); \Gamma)$ such that $\boldsymbol{y}(0) = \boldsymbol{x}$, $\boldsymbol{y}'(0) = \boldsymbol{e}_i$, $|\boldsymbol{y}'(s)| = 1$, $\boldsymbol{y}(s) - \boldsymbol{x} \in \langle \boldsymbol{\nu}(\boldsymbol{x}), \ \boldsymbol{e}_i \rangle$. Then we have

$$\kappa_i \boldsymbol{e}_i = -\frac{d}{ds}[\boldsymbol{\nu}(\boldsymbol{y}(s))]|_{s=0}. \tag{1.3.1}$$

Hence we have

$$\kappa_i \boldsymbol{e}_i = -\left(\nabla_\Gamma^T \boldsymbol{\nu}(\boldsymbol{y}(0)) \right) \boldsymbol{y}'(0) = W(\boldsymbol{x})\boldsymbol{e}_i.$$

Since the eigenvectors of matrix $W(\boldsymbol{x})$ consists of the orthonormal basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{m-1}, \boldsymbol{\nu}(\boldsymbol{x})\}$, it follows that the matrix $W(\boldsymbol{x})$ is symmetric. □

Exercise 1.3.7. *Prove* (1.3.1).

Definition 1.3.8 (Weingarten map). *We consider the linear map correspinding to the matrix* $W(\boldsymbol{x})$

$$\begin{array}{ccccc} W(\boldsymbol{x}) & : & \mathbb{R}^m & \longrightarrow & \mathrm{T}_{\boldsymbol{x}}(\Gamma) \subset \mathbb{R}^m \\ & & \Cup & & \Cup \\ & & \boldsymbol{p} & \longmapsto & \displaystyle\sum_{i=1}^{m-1} \kappa_i (\boldsymbol{p} \cdot \boldsymbol{e}_i) \boldsymbol{e}_i \end{array}$$

which does not depend on the choice of the orthogonal coordinate system. We call it the extended Weingarten map at $\boldsymbol{x} \in \Gamma$*. If we restrict the map* $W(\boldsymbol{x})$ *to the tangent space* $\mathrm{T}_{\boldsymbol{x}}(\Gamma)$*, this is usually called the Weingarten map or the second fundamental tensor.*

The following corollary is clear from the fact that $\{\kappa_1(\boldsymbol{x}), \cdots, \kappa_{m-1}(\boldsymbol{x}), 0\}$ are the eigenvalues of $W(\boldsymbol{x})$.

Corollary 1.3.9.

$$\kappa(\boldsymbol{x}) = \mathrm{tr}\ W(\boldsymbol{x}) = -\mathrm{div}_\Gamma \boldsymbol{\nu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma),$$

$$\kappa_g(\boldsymbol{x}) = \det\ \left(W(\boldsymbol{x}) + \boldsymbol{\nu}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})^{\mathrm{T}}\right) \qquad (\boldsymbol{x} \in \Gamma).$$

Proposition 1.3.10. *If an extension* $\tilde{\boldsymbol{\nu}}$ *of* $\boldsymbol{\nu}$ *satisfies*

$$\tilde{\boldsymbol{\nu}} \in C^1(\mathcal{O}, \mathbb{R}^m), \qquad |\tilde{\boldsymbol{\nu}}(\boldsymbol{x})| = 1 \quad (\boldsymbol{x} \in \mathcal{O}), \qquad \tilde{\boldsymbol{\nu}}|_\Gamma = \boldsymbol{\nu},$$

then $\kappa = -\mathrm{div}\tilde{\boldsymbol{\nu}}$ *holds on* Γ.

Proof. We fix $\boldsymbol{x} \in \Gamma$. For $\rho \in \mathbb{R}$, if $|\rho| << 1$, then

$$|\tilde{\boldsymbol{\nu}}(\boldsymbol{x} - \rho\boldsymbol{\nu}(\boldsymbol{x}))|^2 = 1.$$

Differentiating this equality by ρ at $\rho = 0$, we obtain

$$0 = \frac{d}{d\rho}|\tilde{\boldsymbol{\nu}}(\boldsymbol{x} - \rho\boldsymbol{\nu}(\boldsymbol{x}))|^2\bigg|_{\rho=0} = 2\tilde{\boldsymbol{\nu}}(\boldsymbol{x})^{\mathrm{T}}\ \{\nabla^{\mathrm{T}}\tilde{\boldsymbol{\nu}}(\boldsymbol{x})(-\boldsymbol{\nu}(\boldsymbol{x}))\} = -2\boldsymbol{\nu}(\boldsymbol{x})^{\mathrm{T}}\ (\nabla^{\mathrm{T}}\tilde{\boldsymbol{\nu}}(\boldsymbol{x}))\ \boldsymbol{\nu}(\boldsymbol{x}).$$

Hence, from Proposition 1.3.4, we have

$$\mathrm{div}\tilde{\boldsymbol{\nu}} = \mathrm{div}_\Gamma\tilde{\boldsymbol{\nu}} - \boldsymbol{\nu}^{\mathrm{T}}\ (\nabla^{\mathrm{T}}\tilde{\boldsymbol{\nu}})\ \boldsymbol{\nu} = \mathrm{div}_\Gamma\boldsymbol{\nu} = -\kappa \qquad \text{on} \quad \Gamma.$$

□

Exercise 1.3.11. *If* Γ *is given by a graph* $\eta = u(\boldsymbol{\xi})$ *with* $u \in C^2(\mathbb{R}^{m-1})$*, prove that*

$$\kappa(\boldsymbol{x}) = -\mathrm{div}_\xi \left(\frac{\nabla_\xi u(\boldsymbol{\xi})}{\sqrt{1 + |\nabla_\xi u(\boldsymbol{\xi})|^2}}\right) \qquad \textit{for} \quad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{\xi} \\ u(\boldsymbol{\xi}) \end{pmatrix} \in \Gamma.$$

Lemma 1.3.12. *For $f \in C^1(\Gamma)$ and $\boldsymbol{h} \in C^1(\Gamma, \mathbb{R}^m)$, we have*

$$\mathrm{div}_\Gamma(f\boldsymbol{h}) = f\,\mathrm{div}_\Gamma\boldsymbol{h} + \nabla_\Gamma f \cdot \boldsymbol{h} \qquad \text{on} \quad \Gamma.$$

Exercise 1.3.13. *Prove Lemma 1.3.12.*

Theorem 1.3.14 (Laplacian in curvilinear coordinate). *For $f \in C^2(\mathcal{O})$, we have*

$$\Delta f = \Delta_\Gamma f - \kappa \frac{\partial f}{\partial \boldsymbol{\nu}} + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} \qquad \text{on} \quad \Gamma.$$

In particular, if $\Gamma(r) := \{\boldsymbol{x} \in \mathbb{R}^m;\ |\boldsymbol{x}| = r\}$, then we have the following Laplacian in the polar coordinate

$$\Delta f(\boldsymbol{x}) = \Delta_{\Gamma(r)} f + \frac{m-1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \qquad \text{for} \quad |\boldsymbol{x}| = r.$$

Proof. At a point $\boldsymbol{x} \in \Gamma$, we have

$$\begin{aligned} \Delta f &= \mathrm{div}(\nabla f) = \mathrm{div}_\Gamma(\nabla f) + \boldsymbol{\nu}^{\mathrm{T}}(\nabla^2 f)\boldsymbol{\nu} = \mathrm{div}_\Gamma\left(\nabla_\Gamma f + \boldsymbol{\nu}\frac{\partial f}{\partial \boldsymbol{\nu}}\right) + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} \\ &= \Delta_\Gamma f + (\mathrm{div}_\Gamma \boldsymbol{\nu})\frac{\partial f}{\partial \boldsymbol{\nu}} + \boldsymbol{\nu}^{\mathrm{T}}\nabla_\Gamma\left(\frac{\partial f}{\partial \boldsymbol{\nu}}\right) + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} = \Delta_\Gamma f + -\kappa\frac{\partial f}{\partial \boldsymbol{\nu}} + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2}. \end{aligned}$$

Theorem 1.3.15 (Gauss-Green formula on Γ). *Let $\boldsymbol{h} \in C^1(\Gamma, \mathbb{R}^m)$ and $f \in C^1(\Gamma)$. We assume that $\mathrm{supp}(\boldsymbol{h}f)$ is compact.* [1] *Then we have*

$$\int_\Gamma \boldsymbol{h} \cdot \nabla_\Gamma f d\mathcal{H}^{m-1} = -\int_\Gamma (\mathrm{div}_\Gamma \boldsymbol{h} + \kappa \boldsymbol{\nu} \cdot \boldsymbol{h}) f d\mathcal{H}^{m-1}.$$

A proof of this theorem will be given in the next section.

Corollary 1.3.16. *For $f,\ g \in C^1(\Gamma)$ with compact $\mathrm{supp}(fg)$, we have*

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g \ d\mathcal{H}^{m-1} = -\int_\Gamma f \ \Delta_\Gamma g \ d\mathcal{H}^{m-1}.$$

[1] If Γ is compact, this condition is always satisfied.