Qualitative and quantitative aspects of curvature driven flows of planar curves

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Abstract. In this lecture notes we are concerned with evolution of plane curves satisfying a geometric equation $v = \beta(k, x, \nu)$ where v is the normal velocity of an evolving family of planar closed curves. We assume the normal velocity to be a function of the curvature k, tangential angle ν and the position vector x of a plane curve Γ. We follow the direct approach and we analyze the so-called intrinsic heat equation governing the motion of plane curves obeying such a geometric equation. We show how to reduce the geometric problem to a solution of fully nonlinear parabolic equation for important geometric quantities. Using a theory of fully nonlinear parabolic equations we present results on local in time existence of classical solutions. We also present an approach based on level set representation of curves evolved by the curvature. We recall basic ideas from the theory of viscosity solutions for the level set equation. We discuss numerical approximation schemes for computing curvature driven flows and we present various examples of application of theoretical results in practical problems.

Contents

Preface

The lecture notes on Qualitative and quantitative aspects of curvature driven flows of plane curves are based on the series of lectures given by the author in the fall of 2006 during his stay at the Nečas Center for Mathematical Modeling at Charles University in Prague. The principal goal was to present basic facts and known results in this field to PhD students and young researchers of NCMM.

The main purpose of these notes is to present theoretical and practical topics in the field of curvature driven flows of planar curves and interfaces. There are many recent books and lecture notes on this topic. My intention was to find a balance between presentation of subtle mathematical and technical details and ability of the material to give a comprehensive overview of possible applications in this field. This is often a hard task but I tried to find this balance.

I am deeply indebted to Karol Mikula for long and fruitful collaboration on the problems of curvature driven flows of curves. A lot of the material presented in these lecture notes has been jointly published with him. I want to acknowledge a recent collaboration with V. Srikrishnan who brought to my attention important problems arising in tracking of moving boundaries. I also wish to thank Josef Málek from NCMM of Charles University in Prague for giving me a possibility to visit NCMM and present series of lectures and for his permanent encouragement to prepare these lecture notes.

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CHAPTER 1

Introduction

In this lecture notes we are concerned with evolution of plane curves satisfying a geometric equation

$$v = \beta(k, x, \nu) \tag{1.1}$$

where v is the normal velocity of an evolving family of planar closed curves. We assume the normal velocity to be a function of the curvature k, tangent angle ν and the position vector x of a plane curve Γ.

Geometric equations of the form (1.1) can be often found in variety of applied problems like e.g. the material science, dynamics of phase boundaries in thermomechanics, in modeling of flame front propagation, in combustion, in computations of first arrival times of seismic waves, in computational geometry, robotics, semiconductors industry, etc. They also have a special conceptual importance in image processing and computer vision theories. A typical case in which the normal velocity v may depend on the position vector x can be found in image segmentation [**CKS97, KKO⁺96**]. For a comprehensive overview of other important applications of the geometric Eq. (1.1) we refer to recent books by Sethian, Sapiro and Osher and Fedkiw [**Set96, Sap01, OF03**].

1.1. Mathematical models leading to curvature driven flows of planar curves

1.1.1. Interface dynamics. If a solid phase occupies a subset $\Omega_s(t) \subset \Omega$ and a liquid phase - a subset $\Omega_l(t) \subset \Omega$, $\Omega \subset \mathbb{R}^2$, at a time t, then the phase interface is the set $\Gamma^t = \partial\Omega_s(t) \cap \partial\Omega_l(t)$ which is assumed to be a closed smooth embedded curve. The sharp-interface description of the solidification process is then described by the Stefan problem with a surface tension

$$\rho c \partial_t U = \lambda \Delta U \qquad \text{in } \Omega_s(t) \text{ and } \Omega_l(t),$$

$$[\lambda \partial_n U]_s^l = -Lv \qquad \text{on } \Gamma^t, \tag{1.2}$$

$$\frac{\delta e}{\sigma}(U - U^*) = -\delta_2(\nu)k + \delta_1(\nu)v \text{ on } \Gamma^t, \tag{1.3}$$

subject to initial and boundary conditions for the temperature field U and initial position of the interface Γ (see e.g. [**Ben01**]). The constants ρ, c, λ represent material characteristics (density, specific heat and thermal conductivity), L is the latent heat per unit volume, U^* is a melting point and v is the normal velocity of an interface. Discontinuity in the heat flux on the interface Γ^t is described by the Stefan condition (1.2). The relationship (1.3) is referred to as the Gibbs – Thomson law on the interface Γ^t, where δe is difference in entropy per unit volume between liquid and solid phases, σ is a constant surface tension, δ_1 is a coefficient of attachment kinetics and dimensionless function δ_2 describes anisotropy of the interface. One can see that the Gibbs–Thomson condition can be viewed as a geometric equation having the form (1.1). In this application the normal velocity $v = \beta(k, x, \nu)$ has a special form

$$\beta = \beta(k, \nu) = \delta(\nu)k + F$$

In the theory of phase interfaces separating solid and liquid phases, the geometric equation (1.1) with $\beta(k, x, \nu) = \delta(\nu)k + F$ corresponds to the so-called Gibbs–Thomson law governing the crystal growth in an undercooled liquid [**Gur93, BM98**]. In the series of papers [**AG89, AG94, AG96**]. Angenent and Gurtin studied motion of phase interfaces. They proposed to study the equation of the form

$$\mu(\nu, v)v = h(\nu)k - g$$

where μ is the kinetic coefficient and quantities h, g arise from constitutive description of the phase boundary. The dependence of the normal velocity v on the curvature k is related to surface tension effects on the interface, whereas the dependence on ν (orientation of interface) introduces anisotropic effects into the model. In general, the kinetic coefficient μ may also depend on the velocity v itself giving rise to a nonlinear dependence of the function $v = \beta(k, \nu)$ on k and ν. If the motion of an interface is very slow, then $\beta(k, x, \nu)$ is linear in k (cf. [**AG89**]) and (1.1) corresponds to the classical mean curvature flow studied extensively from both the mathematical (see, e.g., [**GH86, AL86, Ang90a, Gra87**]) and numerical point of view (see, e.g., [**Dzi94, Dec97, MK96, NPV93, OS88**]).

In the series of papers [**AG89, AG96**], Angenent and Gurtin studied perfect conductors where the problem can be reduced to a single equation on the interface. Following their approach and assuming a constant kinetic coefficient one obtains the equation

$$v = \beta(k, \nu) \equiv \delta(\nu)k + F$$

describing the interface dynamics. It is often referred to as the *anisotropic curve shortening equation* with a constant driving force F (energy difference between bulk phases) and a given anisotropy function δ.

1.1.2. Image segmentation. A similar equation to (1.1) arises from the theory of image segmentation in which detection of object boundaries in the analyzed image plays an important role. A given black and white image can be represented by its intensity function $I : R^2 \to [0, 255]$. The aim is to detect edges of the image, i.e. closed planar curves on which the gradient ∇I is large (see [**KM95**]). The idea behind the so-called *active contour models* is to construct an evolving family of plane curves converging to an edge (see [**KWT87**]). One can construct a family of curves evolved by the normal velocity $v = \beta(k, x, \nu)$ of the form

$$\beta(k, x, \nu) = \delta(x, \nu)k + c(x, \nu)$$

where $c(x, \nu)$ is a driving force and $\delta(x, \nu) > 0$ is a smoothing coefficient. These functions depend on the position vector x as well as orientation angle ν of a curve. Evolution starts from an initial curve which is a suitable approximation of the edge and then it converges to the edge provided that δ, c are suitable chosen functions.

In the context of level set methods, edge detection techniques based on this idea were first discussed by Caselles et al. and Malladi et al. in [**CCCD93, MSV95**]. Later on, they have been revisited and improved in [**CKS97, CKSS97, KKO$^+$96**].

1.1.3. Geodesic curvature driven flow of curves on a surface. Another interesting application of the geometric equation (1.1) arises from the differential geometry. The purpose is to investigate evolution of curves on a given surface driven by the geodesic curvature and prescribed external force. We restrict our attention to the case when the normal velocity $\mathcal{V}$ is a linear function of the geodesic curvature $\mathcal{K}_g$ and external force $\mathcal{F}$, i.e. $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ and the surface $\mathcal{M}$ in $\mathbb{R}^3$ can be represented by a smooth graph. The idea how to analyze a flow of curves on a surface $\mathcal{M}$ consists in vertical projection of surface curves into the plane. This allows for reducing the problem to the analysis of evolution of planar curves instead of surface ones. Although the geometric equation $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ is simple the description of the normal velocity v of the family of projected planar curves is rather involved. Nevertheless, it can be written in the form of equation (1.1). The precise form of the function β can be found in the last section.

1.2. Methodology

Our methodology how to solve (1.1) is based on the so-called direct approach investigated by Dziuk, Deckelnick, Gage and Hamilton, Grayson, Mikula and Ševčovič and other authors (see e.g. [**Dec97, Dzi94, Dzi99, GH86, Gra87, MK96, Mik97, MS99, MS01, MS04a, MS04b**] and references therein). The main idea is to use the so-called Lagrangean description of motion and to represent the flow of planar curves by a position vector x which is a solution to the geometric equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}$$

where $\vec{N}, \vec{T}$ are the unit inward normal and tangent vectors, resp. It turns out that one can construct a closed system of parabolic-ordinary differential equations for relevant geometric quantities: the curvature, tangential angle, local length and position vector. Other well-known techniques, like e.g. level-set method due to Osher and Sethian [**Set96, OF03**] or phase-field approximations (see e.g. Caginalp, Nochetto et al., Beneš [**Cag90, NPV93, Ben01**]) treat the geometric equation (1.1) by means of a solution to a higher dimensional parabolic problem. In comparison to these methods, in the direct approach one space dimensional evolutionary problems are solved only. Notice that the direct approach for solving (1.1) can be accompanied by a proper choice of tangential velocity α significantly improving and stabilizing numerical computations as it was documented by many authors (see e.g. [**Dec97, HLS94, HKS98, Kim97, MS99, MS01, MS04a, MS04b**]).

1.3. Numerical techniques

Analytical methods for mathematical treatment of (1.1) are strongly related to numerical techniques for computing curve evolutions. In the *direct approach* one seeks for a parameterization of the evolving family of curves. By solving the so-called *intrinsic heat equation* one can directly find a position vector of a curve (see e.g. [**Dzi91, Dzi94, Dzi99, MS99, MS01, MS04a**]). There are also other direct methods based on solution of a porous medium–like equation for curvature of a curve [**MK96, Mik97**], a crystalline curvature approximation [**Gir95, GK94, UY00**], special finite difference schemes [**Kim94, Kim97**], and a method based on erosion of polygons in the affine invariant scale case [**Moi98**]. By contrast to the direct approach, *level set methods* are based on introducing an auxiliary function whose zero level sets represent an evolving family of planar curves undergoing the geometric equation (1.1) (see, e.g., [**OS88, Set90, Set96, Set98, HMS98**]). The other indirect method is based on the phase-field formulations (see, e.g., [**Cag90, NPV93, EPS96, BM98**]). The level set approach handles implicitly the curvature-driven motion, passing the problem to higher dimensional space. One can deal with splitting and/or merging of evolving curves in a robust way. However, from the computational point of view, level set methods are much more expensive than methods based on the direct approach.

CHAPTER 2

Preliminaries

The purpose of this section is to review basic facts and results concerning a curvature driven flow of planar curves. We will focus our attention on the so-called Langrangean description of a moving curve in which we follow an evolution of point positions of a curve. This is also referred to as a direct approach in the context of curvature driven flows of planar curves ([**AL86, Dzi91, Dzi94, Dec97, MK96, MS99, MS01**]).

First we recall some basic facts and elements of differential geometry. Then we derive a system of equations for important geometric quantities like e.g. a curvature, local length and tangential angle. With help of these equations we shall be able to derive equations describing evolution of the total length, enclosed area of an evolving curve and transport of a scalar function quantity.

2.1. Notations and elements of differential geometry

An embedded regular plane curve (a Jordan curve) Γ is a closed C^1 smooth one dimensional nonselfintersecting curve in the plane $\mathbb{R}^2$. It can be parameterized by a smooth function $x : S^1 \to \mathbb{R}^2$. It means that $\Gamma = \mathrm{Img}(x) := \{x(u), u \in S^1\}$ and $g = |\partial_u x| > 0$. Taking into account the periodic boundary conditions at $u = 0, 1$ we can hereafter identify the unit circle S^1 with the interval $[0, 1]$. The unit arc-length parameterization of a curve $\Gamma = \mathrm{Img}(x)$ is denoted by s and it satisfies $|\partial_s x(s)| = 1$ for any s. Furthermore, the arc-length parameterization is related to the original parameterization u via the equality $\mathrm{d}s = g\,\mathrm{d}u$. Notice that the interval of values of the arc-length parameter depends on the curve Γ. More precisely, $s \in [0, L(\Gamma)]$ where $L(\Gamma)]$ is the length of the curve Γ. Since s is the arc-length parameterization the tangent vector $\vec{T}$ of a curve Γ is given by $\vec{T} = \partial_s x = g^{-1}\partial_u x$. We choose orientation of the unit inward normal vector $\vec{N}$ in such a way that $\det(\vec{T}, \vec{N}) = 1$ where $\det(\vec{a}, \vec{b})$ is the determinant of the 2×2 matrix with column vectors $\vec{a}, \vec{b}$. Notice that $1 = |\vec{T}|^2 = (\vec{T}.\vec{T})$. Therefore, $0 = \partial_s(\vec{T}.\vec{T}) = 2(\vec{T}.\partial_s\vec{T})$. Here $a.b$ denotes the standard Euclidean scalar product in $\mathbb{R}^2$. Thus the direction of the normal vector $\vec{N}$ must be proportional to $\partial_s\vec{T}$. It means that there is a real number $k \in \mathbb{R}$ such that $\vec{N} = k\partial_s\vec{T}$. Similarly, as $1 = |\vec{N}|^2 = (\vec{N}.\vec{N})$ we have $0 = \partial_s(\vec{N}.\vec{N}) = 2(\vec{N}.\partial_s\vec{N})$ and so $\partial_s\vec{N}$ is collinear to the vector $\vec{T}$. Since $(\vec{N}.\vec{T}) = 0$ we have $0 = \partial_s(\vec{N}.\vec{T}) = (\partial_s\vec{N}.\vec{T}) + (\vec{N}.\partial_s\vec{T})$. Therefore, $\partial_s\vec{N} = -k\vec{T}$. In summary, for the arc-length derivative of the unit tangent and normal vectors to a curve Γ we have

$$\partial_s\vec{T} = k\vec{N}, \qquad \partial_s\vec{N} = -k\vec{T} \tag{2.1}$$

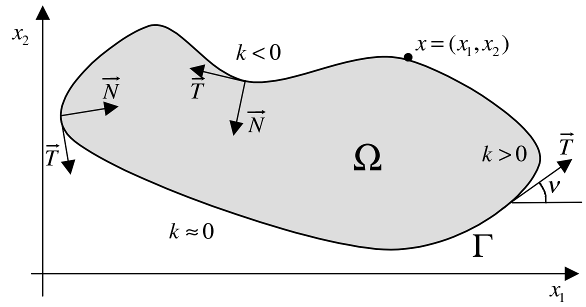

FIGURE 1. Description of a planar curve Γ enclosing a domain Ω, its signed curvature k, unit inward normal $\vec{N}$ and tangent vector $\vec{T}$, position vector x.

where the scalar quantity $k \in \mathbb{R}$ is called a curvature of the curve Γ at a point $x(s)$. Equations (2.1) are referred to as Frenét formulae. The quantity k obeying (2.1) is indeed a curvature in the sense that it is a reciprocal value of the radius of an oscullating circle having C^2 smooth contact with Γ point at a point $x(s)$. Since $\partial_s \vec{T} = \partial_s^2 x$ we obtain a formula for the signed curvature

$$k = \det(\partial_s x, \partial_s^2 x) = g^{-3} \det(\partial_u x, \partial_u^2 x)\,. \tag{2.2}$$

Notice that, according to our notation, the curvature k is positive on the convex side of a curve Γ whereas it is negative on its concave part (see Fig. 1). By ν we denote the tangent angle to Γ, i.e. $\nu = \arg(\vec{T})$ and $\vec{T} = (\cos\nu, \sin\nu)$. Then, by Frenét's formulae, we have

$$k(-\sin\nu, \cos\nu) = k\vec{N} = \partial_s \vec{T} = \partial_s \nu(-\sin\nu, \cos\nu)$$

and therefore

$$\partial_s \nu = k\,.$$

For an embedded planar curve Γ, its tangential angle ν varies from 0 to 2π and so we have $2\pi = \nu(1) - \nu(0) = \int_0^1 \partial_u \nu \,\mathrm{d}u = \int_0^1 kg \,\mathrm{d}u = \int_\Gamma k \,\mathrm{d}s$ and hence the total curvature of an embedded curve satisfies the following equality:

$$\int_\Gamma k \mathrm{d}s = 2\pi\,. \tag{2.3}$$

We remind ourselves that the above equality can be generalized to the case when a closed nonselfintersecting smooth curve Γ belongs to an orientable two dimensional surface $\mathcal{M}$. According to the Gauss-Bonnet formula we have

$$\int_{int(\Gamma)} K \mathrm{d}x + \int_\Gamma k \,\mathrm{d}s = 2\pi\,\chi(\mathcal{M})$$

where K is the Gaussian curvature of am orientable two dimensional surface $\mathcal{M}$ and $\chi(\mathcal{M})$ is the Euler characteristics of the surface $\mathcal{M}$. In a trivial case when $\mathcal{M} = \mathbb{R}^2$ we have $K \equiv 0$ and $\chi(\mathcal{M}) = 1$ and so the equality (2.3) is a consequence of the Gauss-Bonnet formula.

2.2. Governing equations

In these lecture notes we shall assume that the normal velocity v of an evolving family of plane curves $\Gamma^t, t \geq 0$, is equal to a function β of the curvature k, tangential angle ν and position vector $x \in \Gamma^t$,

$$v = \beta(x, k, \nu)\,.$$

(see (1.1). Hereafter, we shall suppose that the function $\beta(k, x, \nu)$ is a smooth function which is increasing in the k variable, i.e.

$$\beta'_k(k, x, \nu) > 0\,.$$

An idea behind the direct approach consists of representation of a family of embedded curves Γ^t by the position vector $x \in \mathbb{R}^2$, i.e.

$$\Gamma^t = \mathrm{Img}(x(., t)) = \{x(u, t),\ u \in [0, 1]\}$$

where x is a solution to the geometric equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \tag{2.4}$$

where $\beta = \beta(x, k, \nu)$, $\vec{N} = (-\sin\nu, \cos\nu)$ and $\vec{T} = (\cos\nu, \sin\nu)$ are the unit inward normal and tangent vectors, respectively. For the normal velocity $v = \partial_t x.\vec{N}$ we have $v = \beta(x, k, \nu)$. Notice that the presence of arbirary tangential velocity functional α has no impact on the shape of evolving curves.

The goal of this section is to derive a system of PDEs governing the evolution of the curvature k of $\Gamma^t = \mathrm{Img}(x(., t)), t \in [0, T)$, and some other geometric quantities where the family of regular plane curves where $x = x(u, t)$ is a solution to the position vector equation (2.4). These equations will be used in order to derive a priori estimates of solutions. Notice that such an equation for the curvature is well known for the case when $\alpha = 0$, and it reads as follows: $\partial_t k = \partial_s^2 \beta + k^2 \beta$ (cf. [**GH86, AG89**]). Here we present a brief sketch of the derivation of the corresponding equations for the case of a nontrivial tangential velocity α.

Let us denote $\vec{p} = \partial_u x$. Since $u \in [0, 1]$ is a fixed domain parameter we commutation relation $\partial_t \partial_u = \partial_u \partial_t$. Then, by using Frenét's formulae, we obtain

$$\begin{aligned}
\partial_t \vec{p} &= |\partial_u x|((\partial_s \beta + \alpha k)\vec{N} + (-\beta k + \partial_s \alpha)\vec{T}),\\
\vec{p}\,.\,\partial_t \vec{p} &= |\partial_u x|\ \vec{T}\,.\,\partial_t \vec{p} = |\partial_u x|^2(-\beta k + \partial_s \alpha),\\
\det(\vec{p}, \partial_t \vec{p}) &= |\partial_u x|\ \det(\vec{T}, \partial_t \vec{p}) = |\partial_u x|^2\ (\partial_s \beta + \alpha k),\\
\det(\partial_t \vec{p}, \partial_u \vec{p}) &= -|\partial_u x| \partial_u |\partial_u x| (\partial_s \beta + |\partial_u x|^3\ (-\beta k + \partial_s \alpha),
\end{aligned} \tag{2.5}$$

because $\partial_u \vec{p} = \partial_u^2 x = \partial_u(|\partial_u x|\ \vec{T}) = \partial_u |\partial_u x|\ \vec{T} + k|\partial_u x|^2\ \vec{N}$. Since $\partial_u \det(\vec{p}, \partial_t \vec{p}) = \det(\partial_u \vec{p}, \partial_t \vec{p}) + \det(\vec{p}, \partial_u \partial_t \vec{p})$, we have $\det(\vec{p}, \partial_u \partial_t \vec{p}) = \partial_u \det(\vec{p}, \partial_t \vec{p}) + \det(\partial_t \vec{p}, \partial_u \vec{p})$. As $k = \det(\vec{p}, \partial_u \vec{p})\, |\vec{p}|^{-3}$ (see (2.2)), we obtain

$$\begin{aligned}
\partial_t k &= -3|p|^{-5}(\vec{p}\,.\,\partial_t \vec{p}) \det(\vec{p}, \partial_u \vec{p}) + |\vec{p}|^{-3}\,(\det(\partial_t \vec{p}, \partial_u \vec{p}) + \det(\vec{p}, \partial_u \partial_t \vec{p}))\\
&= -3k|\vec{p}|^{-2}(\vec{p}\,.\,\partial_t \vec{p}) + 2|\vec{p}|^{-3} \det(\partial_t \vec{p}, \partial_u \vec{p}) + |\vec{p}|^{-3} \partial_u \det(\vec{p}, \partial_t \vec{p}).
\end{aligned}$$

Finally, by applying identities (2.5), we end up with the second-order nonlinear parabolic equation for the curvature:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta\,. \tag{2.6}$$

The identities (2.5) can be used in order to derive an evolutionary equation for the local length $|\partial_u x|$. Indeed, $\partial_t|\partial_u x| = (\partial_u x\,.\,\partial_u\partial_t x)/|\partial_u x| = (\vec{p}\,.\,\partial_t\vec{p})/|\partial_u x|$. By (2.5) we have the

$$\partial_t|\partial_u x| = -|\partial_u x|\,k\beta \ + \partial_u\alpha \tag{2.7}$$

where $(u,t) \in Q_T = [0,1]\times[0,T)$. In other words, $\partial_t \mathrm{d}s = (-k\beta + \partial_s\alpha)\mathrm{d}s$. It yields the commutation relation

$$\partial_t\partial_s - \partial_s\partial_t = (k\beta - \partial_s\alpha)\partial_s. \tag{2.8}$$

Next we derive equations for the time derivative of the unit tangent vector $\vec{T}$ and tangent angle ν. Using the above commutation relation and Frenét formulae we obtain

$$\begin{aligned}
\partial_t\vec{T} &= \partial_t\partial_s x = \partial_s\partial_t x + (k\beta - \partial_s\alpha)\partial_s x\,,\\
&= \partial_s(\beta\vec{N} + \alpha\vec{T}) + (k\beta - \partial_s\alpha)\vec{T}\,,\\
&= (\partial_s\beta + \alpha k)\vec{N}\,.
\end{aligned}$$

Since $\vec{T} = (\cos\nu, \sin\nu)$ and $\vec{N} = (-\sin\nu, \cos\nu)$ we conclude that $\partial_t\nu = \partial_s\beta + \alpha k$. Summarizing, we end up with evolutionary equations for the unit tangent and normal vectors $\vec{T}, \vec{N}$ and the tangent angle ν

$$\begin{aligned}
\partial_t\vec{T} &= (\partial_s\beta + \alpha k)\vec{N}\,,\\
\partial_t\vec{N} &= -(\partial_s\beta + \alpha k)\vec{T}\,,\\
\partial_t\nu &= \partial_s\beta + \alpha k\,.
\end{aligned} \tag{2.9}$$

Since $\partial_s\nu = k$ and $\partial_s\beta = \beta'_k\partial_s k + \beta'_\nu k + \nabla_x\beta.\vec{T}$ we obtain the following closed system of parabolic-ordinary differential equations:

$$\partial_t k = \partial_s^2\beta + \alpha\partial_s k + k^2\beta\,, \tag{2.10}$$

$$\partial_t\nu = \beta'_k\partial_s^2\nu + (\alpha + \beta'_\nu)\partial_s\nu + \nabla_x\beta.\vec{T}\,, \tag{2.11}$$

$$\partial_t g = -gk\beta + \partial_u\alpha\,, \tag{2.12}$$

$$\partial_t x = \beta\vec{N} + \alpha\vec{T}\,, \tag{2.13}$$

where $(u,t) \in Q_T = [0,1]\times(0,T)$, $\mathrm{d}s = g\,\mathrm{d}u$ and $\vec{T} = \partial_s x = (\cos\nu, \sin\nu), \vec{N} = \vec{T}^\perp = (-\sin\nu,\ \cos\nu)$. The functional α may depend on the variables k, ν, g, x. A solution (k,ν,g,x) to (2.10) – (2.13) is subject to initial conditions

$$k(.,0) = k_0\,,\quad \nu(.,0) = \nu_0\,,\quad g(.,0) = g_0\,,\quad x(.,0) = x_0(.)\,,$$

and periodic boundary conditions at $u = 0, 1$ except of the tangent angle ν for which we require that the tangent vector $\vec{T}(u,t) = (\cos(\nu(u,t)), \sin(\nu(u,t)))$ is 1-periodic in the u variable, i.e. $\nu(1,t) = \nu(0,t) + 2\pi$. Notice that the initial conditions for k_0, ν_0, g_0 and x_0 (the curvature, tangent angle, local length element and position vector of the initial curve Γ_0) must satisfy the following compatibility constraints:

$$g_0 = |\partial_u x_0| > 0\,,\quad k_0 = g_0^{-3}\partial_u x_0 \wedge \partial_u^2 x_0\,,\quad \partial_u\nu_0 = g_0 k_0\,.$$

2.3. First integrals for geometric quantities

The aim of this section is to derive basic identities for various geometric quantities like e.g. the length of a closed curve and the area enclosed by a Jordan curve in the plane. These identities (first integrals) will be used later in the analysis of the governing system of equations.

2.3.1. The total length equation. By integrating (2.7) over the interval $[0,1]$ and taking into account that α satisfies periodic boundary conditions, we obtain the total length equation

$$\frac{\mathrm{d}}{\mathrm{d}t}L^t + \int_{\Gamma^t} k\beta \mathrm{d}s = 0, \tag{2.14}$$

where $L^t = L(\Gamma^t)$ is the total length of the curve Γ^t, $L^t = \int_{\Gamma^t} \mathrm{d}s = \int_0^1 |\partial_u x(u,t)|\,\mathrm{d}u$. If $k\beta \geq 0$, then the evolution of planar curves parameterized by a solution of (1.1) represents a curve shortening flow, i.e., $L^{t_2} \leq L^{t_1} \leq L^0$ for any $0 \leq t_1 \leq t_2 \leq T$. The condition $k\beta \geq 0$ is obviously satisfied in the case $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, where $m > 0$ and γ is a nonnegative anisotropy function. In particular, the Euclidean curvature driven flow ($\beta = k$) is curve shortening flow.

2.3.2. The area equation. Let us denote by $A = A^t$ the area of the domain Ω^t enclosed by a Jordan curve Γ^t. Then by using Green's formula we obtain, for $P = -x_2/2, Q = x_1/2$,

$$A^t = \iint_{\Omega_t} \mathrm{d}x = \iint_{\Omega_t} \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\mathrm{d}x = \oint_{\Gamma^t} P\mathrm{d}x_1 + Q\mathrm{d}x_2 = \frac{1}{2}\oint_{\Gamma^t} -x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2\,.$$

Since $\mathrm{d}x_i = \partial_u x_i \mathrm{d}u, u \in [0,1]$, we have

$$A^t = \frac{1}{2}\int_0^1 \det(x, \partial_u x)\,\mathrm{d}u\,.$$

Clearly, integration of the derivative of a quantity along a closed curve yields zero. Therefore $0 = \int_0^1 \partial_u \det(x, \partial_t x)\mathrm{d}u = \int_0^1 \det(\partial_u x, \partial_t x) + \det(x, \partial_u\partial_t x)\mathrm{d}u$, and so $\int_0^1 \det(x, \partial_u \partial_t x)\mathrm{d}u = \int_0^1 \det(\partial_t x, \partial_u x)\mathrm{d}u$ because $\det(\partial_u x, \partial_t x) = -\det(\partial_t x, \partial_u x)$. As $\partial_t x = \beta\vec{N} + \alpha\vec{T}$, $\partial_u x \mathrm{d}u = \vec{T}\mathrm{d}s$ and $\frac{\mathrm{d}}{\mathrm{d}t}A^t = \frac{1}{2}\int_0^1 2\det(\partial_t x, \partial_u x)\mathrm{d}u$ we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}A^t + \int_{\Gamma^t} \beta \mathrm{d}s = 0. \tag{2.15}$$

Remark. In the case when a curve is evolved according to the curvature, i.e. $\beta = k$, then it follows from (2.3) and (2.15) that $\frac{\mathrm{d}}{\mathrm{d}t}A^t = -2\pi$ and so

$$A^t = A^0 - 2\pi t\,.$$

It means that the curve Γ^t ceases to exists for $t = T_{max} = \frac{A^0}{2\pi}$, i.e. the lifespan of curve evolution with $\beta = k$ is finite.

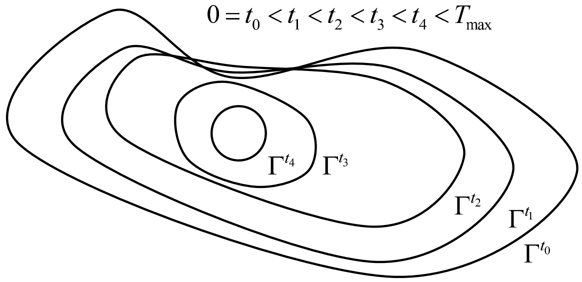

FIGURE 2. A closed curve evolving by the curvature becomes convex in finite time and then it converges to a point.

2.3.3. Brakke's motion by curvature. The above first integrals can be generalized for computation of the time derivative of the quantity $\int_{\Gamma^t} \phi(x,t)\,\mathrm{d}s$ where $\phi \in C_0^\infty(\mathbb{R}^2,[0,T))$ is a compactly supported test function. It represents a total value of a transported quantity represented by a scalar function ϕ. Since the value of the geometric quantity $\int_{\Gamma^t} \phi(x,t)\,\mathrm{d}s$ is independent of a particular choice of a tangential velocity α we may take $\alpha = 0$ for simplicity. Since $\partial_t x = \beta \vec{N}$ and $\partial_t \mathrm{d}s = \partial_t g \mathrm{d}u = -k\beta g \mathrm{d}u = -k\beta \mathrm{d}s$ we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma^t} \phi(x,t)\,\mathrm{d}s &= \int_{\Gamma^t} \partial_t\phi(x,t) + \nabla_x\phi.\partial_t x - k\beta\phi\,\mathrm{d}s \\
&= \int_{\Gamma^t} \partial_t\phi(x,t) + \beta\nabla_x\phi.\vec{N} - k\beta\phi\,\mathrm{d}s\,. \qquad (2.16)
\end{aligned}
$$

The above integral identity (2.16) can be used in description of a more general flow of rectifiable subsets of $\mathbb{R}^2$ with a distributional notion of a curvature which is refereed to as varifold. Let $\Gamma^t, t \in [0,T)$, be a flow of one dimensional countably rectifiable subsets of the plane $\mathbb{R}^2$. Brakke in [**Bra78**, Section 3.3] introduced a notion of a mean curvature flow (i.e. $\beta = k$) as a solution to the following integral inequality

$$
\overline{\frac{\mathrm{d}}{\mathrm{d}t}}\int_{\Gamma^t} \phi(x,t)\,\mathrm{d}\mathcal{H}^1(x) \le \int_{\Gamma^t} \left(\partial_t\phi(x,t) + k\nabla_x\phi.\vec{N} - k^2\phi\right)\,\mathrm{d}\mathcal{H}^1(x) \qquad (2.17)
$$

for any smooth test function $\phi \in C_0^\infty(\mathbb{R}^2,[0,T))$. Here we have denoted by $\overline{\frac{\mathrm{d}}{\mathrm{d}t}}$ the upper derivative and $\mathcal{H}^1(x)$ the one dimensional Hausdorff measure.

2.4. Gage-Hamilton and Grayson's theorems

Assume that a smooth, closed, and embedded curve is evolved along its normal vector at a normal velocity proportional to its curvature, i.e. $\beta = k$. This curve evolution is known as the Euclidean curve shortening flow, and is depicted in Fig. 2. Since the curvature is positive on the convex side and it is negative on the concave side one may expect that the evolving curve becomes more convex and less concave as time t increases. Finally, it becomes convex shape and it shrinks to a circular point in finite time. This natural observation has been rigorously proved by M.

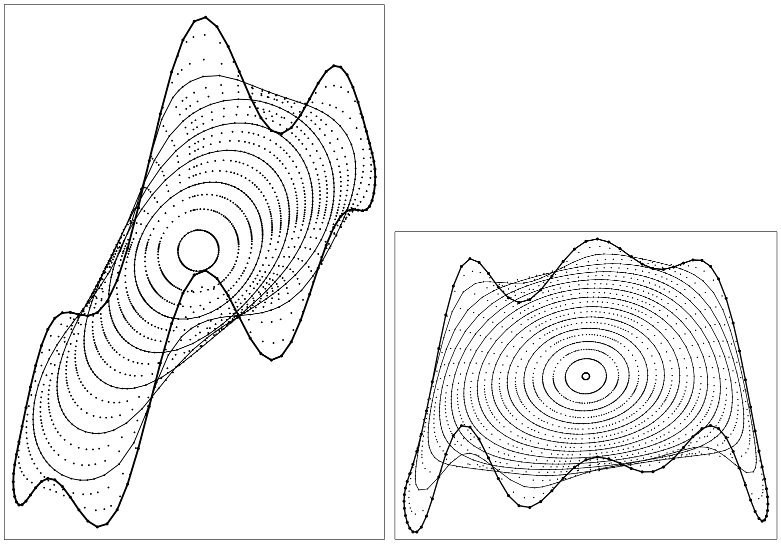

FIGURE 3. Motion by the curvature. Numerically computed evolution of various initial curves.

Grayson in [**Gra87**]. He used already known result due to Gage and Hamilton. They considered evolution of convex curves in the plane and proved that evolved curves shrink to a circular point in finite time.

THEOREM 2.1 (Gage and Hamilton [**GH86**]). *Any smooth closed convex curve embedded in* $\mathbb{R}^2$ *evolved by the curvature converges to a point in finite time with asymptotic circular shape.*

What Grayson added to this proof was the statement that any embedded smooth planar curve (not necessarily convex) when evolving according to the curvature becomes convex in finite time, stays embedded and then it shrinks to a circular point in finite time.

THEOREM 2.2 (Grayson [**Gra87**]). *Any smooth closed curve embedded in* $\mathbb{R}^2$ *evolve by the curvature becomes convex in finite time and then it converges to a point in finite time with asymptotical circular shape.*

Figure 3 shows computational results of curvature driven evolution of two initial planar curve evolved with the normal velocity $\beta = k$.

Although we will not go into the details of proofs of the above theorems it is worthwile to note that the proof of Grayson's theorem consists of several steps. First one needs to prove that an embedded initial curve Γ^0 when evolved according to the curvature stays embedded for $t > 0$, i.e. selfintersections cannot occur for $t > 0$. Then it is necessary to prove that eventual concave parts of a curve decrease they length. To this end, one can construct a partition a curve into its convex and concave part and show that concave parts are vanishing when time increases. The curve eventually becomes convex. Then Grayson applied previous result due to Gage and Hamilton. Their result says that any initial convex curve asymptotically approaches a circle when $t \to T_{max}$ where T_{max} is finite. To interpret their result in the language of parabolic partial differential equations we notice that the solution to (2.10) with $\beta = k$ remains positive provided that the initial value k^0 was nonnegative. This is a direct consequence of the maximum principle for parabolic equations. Indeed, let us denote by $y(t) = \min_{\Gamma^t} k(.,t)$. With regard to the envelope theorem we may assume that there exists $s(t)$ such that $y(t) = k(s(t),t)$. As $\partial_s^2 k \geq 0$ and $\partial_s k = 0$ at $s = s(t)$ we obtain from (2.10) that $y'(t) \geq y^3(t)$. Solving this ordinary differential inequality with positive initial condition $y(0) = \min_{\Gamma^0} k^0 > 0$ we obtain $\min_{\Gamma^t} k(.,t) = y(t) > 0$ for $0 < t < T_{max}$. Thus Γ^t remains convex provided Γ^0 was convex. The proof of the asymptotic circular profile is more complicated. However, it can be very well understood when considering selfsimilarly rescaled dependent and independent variables in equation (2.10). In these new variables, the statement of Gage and Hamilton theorem is equivalent to the proof of asymptotical stability of the constant unit solution.

In the proof of Grayson's theorem one can find another nice application of the parabolic comparison principle. Namely, if one wants to prove embeddednes property of an evolved curve Γ^t it is convenient to inspect the following distance function between arbitrary two points $x(s_1,t), x(s_2,t)$ of a curve Γ^t:

$$f(s_1,s_2,t) = |x(s_1,t) - x(s_2,t)|^2$$

where $s_1, s_2 \in [0, L(\Gamma^t)]$ and $t > 0$. Assume that $x = x(s,t)$ satisfies (2.4). Without loss of generality we may assume $\alpha = 0$ as α does not change the shape of the curve. Hence the embeddednes property is independent of α. Without loss of generality we therefore may choose $\alpha = 0$. Let us compute partial derivatives of f with respect to its variables. With help of Frenét formulae we obtain

$$\begin{aligned}
\partial_t f &= 2((x(s_1,t) - x(s_2,t)).(\partial_t x(s_1,t) - \partial_t x(s_2,t))) \\
&= 2((x(s_1,t) - x(s_2,t)).(k(s_1,t)\vec{N}(s_1,t) - k(s_2,t)\vec{N}(s_2,t))) \\
\partial_{s_1} f &= 2((x(s_1,t) - x(s_2,t)).\vec{T}(s_1,t)) \\
\partial_{s_2} f &= -2((x(s_1,t) - x(s_2,t)).\vec{T}(s_2,t)) \\
\partial_{s_1}^2 f &= 2(\vec{T}(s_1,t).\vec{T}(s_1,t)) + 2k(s_1,t)((x(s_1,t) - x(s_2,t)).\vec{N}(s_1,t)) \\
\partial_{s_2}^2 f &= 2(\vec{T}(s_1,t).\vec{T}(s_1,t)) - 2k(s_2,t)((x(s_1,t) - x(s_2,t).\vec{N}(s_2,t))\,.
\end{aligned}$$

Hence

$$\partial_t f = \Delta f - 4$$

where Δ is the Laplacian operator with respect to variables s_1, s_2. Using a clever application of a suitable barrier function (a circle) and comparison principle for the above parabolic equation Grayson proved that $f(s_1,s_2,t) \geq \delta > 0$ whenever $|s_1 - s_2| \geq \epsilon > 0$ where $\epsilon, \delta > 0$ are sufficiently small. But this is equivalent to the statement that the curve Γ^t is embedded. Notice that the above "trick" works only for the case $\beta = k$ and this is why it is still an open question whether embedded initial curve remains embedded when it is evolved by a general normal velocity $\beta = \beta(k)$.

2.4.1. Asymptotic profile of shrinking curves for other normal velocities. There are some partial results in this direction. If $\beta = k^{1/3}$ then the corresponding flow of planar curves is called affine space scale flow. It has been

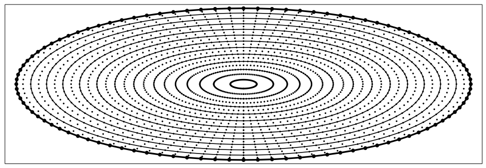

FIGURE 4. An initial ellipse evolved with the normal velocity $\beta = k^{1/3}$.

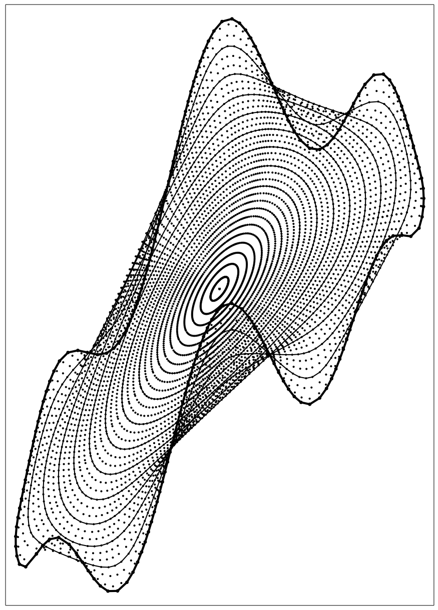

FIGURE 5. An example of evolution of planar curves evolved by the normal velocity $\beta = k^{1/3}$.

studied and analyzed by Angenent, Shapiro and Tannenbaum in [**AST98**] and [**ST94**]. In this case the limiting profile of a shrinking family of curves is an ellipse. Selfsimilar property of shrinking ellipses in the case $\beta = k^{1/3}$ has been also addressed in [**MS99**]. In Fig. 4 we present a computational result of evolution of shrinking ellipses. Fig. 5 depicts evolution of the same initial curve as in Fig. 3 (left) but now the curve is evolved with $\beta = k^{1/3}$. Finally. Fig. 6 shows computational results of curvature driven evolution of an initial spiral-like curve. Notice that the normal velocity of form $\beta(k) = k^\omega$ has been investigated by Ushijima and Yazaki in [**UY00**] in the context of crystaline curvature numerical approximation of the flow. It can be shown that $\omega = 1/3, 1/8, 1/15, ..., 1/(n^2 - 1), ...$, are bifurcation values for which one can prove the existence of branches of selfsimilar solutions of evolving curves shrinking to a point as a rounded polygon with n faces.

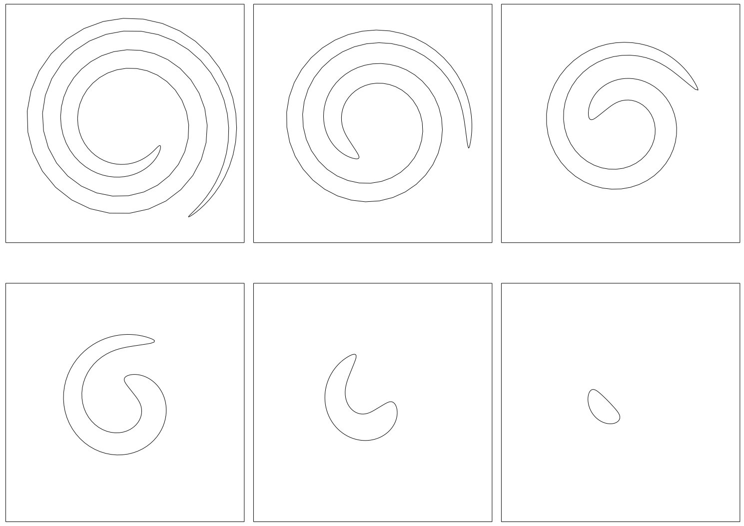

FIGURE 6. The sequence of evolving spirals for $\beta = k^{1/3}$.

CHAPTER 3

Qualitative behavior of solutions

In this chapter we focus our attention on qualitative behavior of curvature driven flows of planar curves. We present techniques how to prove local in time existence of a smooth family of curves evolved with the normal velocity given by a general function $\beta = \beta(k, x, \nu)$ depending on the curvature k, position vector x as well as the tangential angle ν. The main idea is to transform the geometric problem into the language of a time depending solution to an evolutionary partial differential equation like e.g. (2.10)–(2.13). First we present an approach due to Angenent describing evolution of an initial curve by a fully nonlinear parabolic equation for the distance function measuring the normal distance of the initial curve Γ^0 the evolved curve Γ^t for small values of $t > 0$. The second approach presented in this chapter is based on solution to the system of nonlinear parabolic-ordinary differential equations (2.10)–(2.13) also proposed by Angenent and Gurtin [**AG89, AG94**] and further analyzed and applied by Mikula and Ševčovič in the series of papers [**MS01, MS04a, MS04b**]. Both approaches are based on the solution to a certain fully nonlinear parabolic equation or system of equations. To provide a local existence and continuation result we have apply the theory of nonlinear analytic semiflows due to Da Prato and Grisvard, Lunardi [**DPG75, DPG79, Lun82**] and Angenent [**Ang90a, Ang90b**].

3.1. Local existence of smooth solutions

The idea of the proof of a local existence of an evolving family of closed embedded curves is to transform the geometric problem into a solution to a fully nonlinear parabolic equation for the distance $\phi(u,t)$ of a point $x(u,t) \in \Gamma^t$ from its initial value position $x^0(u) = x(u,0) \in \Gamma^0$. This idea is due to Angenent [**Ang90b**] who derived the fully nonlinear parabolic equation for ϕ and proved local existence of smooth solutions by method of abstract nonlinear evolutionary equations in Banach spaces [**Ang90b**].

3.1.1. Local representation of an embedded curve. Let $\Gamma^0 = \mathrm{Img}(x^0)$ be a smooth initial Jordan curve embedded in $\mathbb{R}^2$. Because of its smoothness and embeddednes one can construct a local parameterization of any smooth curve $\Gamma^t = \mathrm{Img}(x(.,t))$ lying in the thin tubular neighborhood along Γ^0, i.e. $\mathrm{dist}_H(\Gamma^t, \Gamma^0) < \varepsilon$ where dist_H is the Hausdorff set distance function. This is why there exists a small number $0 < \varepsilon \ll 1$ and a smooth immersion function $\sigma : S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

- $x^0(u) = \sigma(u, 0)$ for any $u \in S^1$

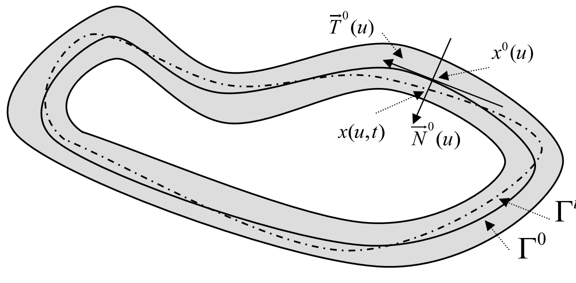

FIGURE 1. Description of a local parameterization of an embedded curve Γ^t in the neighborhood of the initial curve Γ^0.

- for any $u \in S^1$ there exists a unique $\phi = \phi(u,t) \in (-\varepsilon, \varepsilon)$ such that $\sigma(u, \phi(u,t)) = x(u,t)$.
- the implicitly defined function $\phi = \phi(u,t)$ is smooth in its variables provided the function $x = x(u,t)$ is smooth.

It is easy to verify that the function $\sigma(u,\phi) = x^0(u) + \phi \vec{N}^0(u)$ is the immersion having the above properties. Here $\vec{N}^0(u)$ is the unit inward vector to the curve Γ^0 at the point $x^0(u)$ (see Fig. 1).

Now we can evaluate $\partial_t x, \partial_u x, \partial_u^2 x$ and $|\partial_u x|$ as follows:

$$
\begin{aligned}
\partial_t x &= \sigma'_\phi \partial_t \phi\,,\\
\partial_u x &= \sigma'_u + \sigma'_\phi \partial_u \phi\,,\\
\partial_u^2 x &= \sigma''_{uu} + 2\sigma''_{u\phi}\partial_u\phi + \sigma''_{\phi\phi}(\partial_u\phi)^2 + \sigma'_\phi \partial_u^2\phi\,,\\
g = |\partial_u x| &= \left(|\sigma'_u|^2 + 2(\sigma'_u . \sigma'_\phi)\partial_u\phi + |\sigma'_\phi|^2(\partial_u\phi)^2\right)^{\frac{1}{2}}\,.
\end{aligned}
$$

Hence we can express the curvature $k = \det(\partial_u x, \partial_u^2 x)/|\partial_u x|^3$ as follows:

$$
\begin{aligned}
g^3 k &= \det(\partial_u x, \partial_u^2 x) = \partial_u^2\phi\partial_u\phi \det(\sigma'_\phi, \sigma'_\phi) + \partial_u^2\phi \det(\sigma'_u, \sigma'_\phi)\\
&+ (\partial_u\phi)^2 \left[\det(\sigma'_u, \sigma''_{\phi\phi}) + \partial_u\phi \det(\sigma'_\phi, \sigma''_{\phi\phi})\right] + 2\partial_u\phi \det(\sigma'_u, \sigma''_{u\phi})\\
&+ 2(\partial_u\phi)^2 \det(\sigma'_\phi, \sigma''_{u\phi}) + \det(\sigma'_u, \sigma''_{uu}) + \partial_u\phi \det(\sigma'_\phi, \sigma''_{uu})\,.
\end{aligned}
$$

Clearly, $\det(\sigma'_\phi, \sigma'_\phi) = 0$. Since $\sigma'_\phi = \vec{N}^0$ and $\sigma'_u = \partial_u x^0 + \phi\partial_u \vec{N}^0 = g^0(1 - k^0\phi)\vec{T}^0$ we have $\det(\sigma'_u, \sigma'_\phi) = g^0(1 - k^0\phi)$ and $(\sigma'_u . \sigma'_\phi) = 0$. Therefore the local length $g = |\partial_u x|$ and the curvature k can be expressed as

$$
\begin{aligned}
g &= |\partial_u x| = \left((g^0(1 - k^0\phi))^2 + (\partial_u\phi)^2\right)^{\frac{1}{2}}\,,\\
k &= \frac{g^0(1 - k^0\phi)}{g^3}\partial_u^2\phi + R(u, \phi, \partial_u\phi)
\end{aligned}
$$

where $R(u, \phi, \partial_u\phi)$ is a smooth function.

We proceed with evaluation of the time derivative $\partial_t x$. Since $\partial_u x = \sigma'_u + \sigma'_\phi\partial_u\phi$ we have $\vec{T} = \frac{1}{g}(\sigma'_u + \sigma'_\phi\partial_u\phi)$. The vectors $\vec{N}$ and $\vec{T}$ are perpendicular to each other. Thus

$$
\partial_t x . \vec{N} = \det(\partial_t x, \vec{T}) = \frac{1}{g}\det(\sigma'_u, \sigma'_\phi)\partial_t\phi = \frac{g^0(1 - k^0\phi)}{g}\partial_t\phi
$$

because $\det(\sigma'_\phi, \sigma'_\phi) = 0$. Hence, a family of embedded curves $\Gamma^t, t \in [0,T)$, evolves according to the normal velocity

$$\beta = \mu k + c$$

if and only if the function $\phi = \phi(u,t)$ is a solution to the nonlinear parabolic equation

$$\partial_t \phi = \frac{\mu}{g^2}\partial_u^2 \phi + \frac{g}{g^0(1-k^0\phi)}(\mu R(u,\phi,\partial_u\phi) + c)$$

where

$$g = \left(|g^0|^2(1-k^0\phi)^2 + (\partial_u\phi)^2\right)^{\frac{1}{2}} .$$

In a general case when the normal velocity $\beta = \beta(k, x, \vec{N})$ is a function of curvature k, position vector x and the inward unit normal vector $\vec{N}$, ϕ is a solution to a fully nonlinear parabolic equation of the form:

$$\partial_t \phi = F(\partial_u^2\phi, \partial_u\phi, \phi, u), \quad u \in S^1, t \in (0,T)\,. \tag{3.1}$$

The right-hand side function $F = F(q,p,\phi,u)$ is C^1 is a smooth function of its variables and

$$\frac{\partial F}{\partial q} = \frac{\beta'_k}{g^2} > 0$$

and so equation (3.1) is a nonlinear strictly parabolic equation. Equation (3.1) is subject to an initial condition

$$\phi(u,0) = \phi^0(u) \equiv 0\,, \quad u \in S^1\,. \tag{3.2}$$

3.1.2. Nonlinear analytic semiflows. In this section we recall basic facts from the theory of nonlinear analytic semiflows which can be used in order to prove local in time existence of a smooth solutions to the fully nonlinear parabolic equation (3.1) subject to the initial condition (3.2). The theory has been developed by S. Angenent in [**Ang90b**] and A. Lunardi in [**Lun82**].

Equation (3.1) can be rewritten as an abstract evolutionary equation

$$\partial_t \phi = \mathcal{F}(\phi) \tag{3.3}$$

subject to the initial condition

$$\phi(0) = \phi^0 \in E_1 \tag{3.4}$$

where $\mathcal{F}$ is a C^1 smooth mapping between two Banach spaces E_1, E_0, i.e. $\mathcal{F} \in C^1(E_1, E_0)$. For example, if we take

$$E_0 = h^\varrho(S^1), \qquad E_1 = h^{2+\varrho}(S^1)\,,$$

where $h^{k+\varrho}(S^1), k = 0,1,...,$ is a little Hölder space, i.e. the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{k+\sigma}(S^1)$ (see [**Ang90b**]), then the mapping F defined as in the right-hand side of (3.1) is indeed a C^1 mapping from E_1 into E_0. Its Frechét derivative $d\mathcal{F}(\phi^0)$ is being given by the linear operator

$$d\mathcal{F}(\phi^0)\phi = a^0\partial_u^2\phi + b^0\partial_u\phi + c^0\phi$$

where

$$a^0 = F'_q(\partial_u^2\phi^0, \partial_u\phi^0, \phi^0, u) = \frac{\beta'_k}{(g^0)^2}, \quad b^0 = F'_p(\partial_u^2\phi^0, \partial_u\phi^0, \phi^0, u),$$
$$c^0 = F'_\phi(\partial_u^2\phi^0, \partial_u\phi^0, \phi^0, u)\,.$$

Suppose that the initial curve $\Gamma^0 = \mathrm{Img}(x^0)$ is sufficiently smooth, $x^0 \in \left(h^{2+\varrho}(S^1)\right)^2$ and regular, i.e. $g^0(u) = |\partial_u x^0(u)| > 0$ for any $u \in S^1$. Then $a^0 \in h^{1+\varrho}(S^1)$. A standard result from the theory of analytic semigroups (c.f. [**Hen81**]) enables us to conclude that the principal part $A := a^0 \partial_u^2$ of the linearization $d\mathcal{F}(\phi^0)$ is a generator of a analytic semigroup $\exp(tA), t \geq 0$, in the Banach space $E_0 = h^\varrho(S^1)$.

3.1.2.1. *Maximal regularity theory.* In order to proceed with the proof of local in time existence of a classical solution to the abstract nonlinear equation (3.3) we have to recall a notion of a maximal regularity pair of Banach spaces.

Assume that (E_1, E_0) is a pair of Banach spaces with E_1 densely included into E_0. By $L(E_1, E_0)$ we shall denote the Banach space of all linear bounded operators from E_1 into E_0. An operator $A \in L(E_1, E_0)$ can be considered as an unbounded operator in the Banach space E_0 with a dense domain $D(A) = E_1$. By $Hol(E_1, E_0)$ we shall denote a subset of $L(E_1, E_0)$ consisting of all generators A of an analytic semigroup $\exp(tA), t \geq 0$, of linear operators in the Banach space E_0 (c.f. [**Hen81**]).

The next lemma is a standard perturbation result concerning the class of generators of analytic semigroups.

LEMMA 3.1. [**Paz83**, Theorem 2.1] *The set $Hol(E_1, E_0)$ is an open subset of the Banach space $L(E_1, E_0)$.*

The next result is also related to the perturbation theory for the class of generators of analytic semigroups.

DEFINITION 3.2. We say that the linear bounded operator $B : E_1 \to E_0$ has a relative zero norm if for any $\varepsilon > 0$ there is a constant $k_\varepsilon > 0$ such that

$$\|Bx\|_{E_0} \leq \varepsilon \|x\|_{E_1} + k_\varepsilon \|x\|_{E_0}$$

for any $x \in E_1$.

As an example of such an operator we may consider an operator $B \in L(E_1, E_0)$ satisfying the following inequality of Gagliardo-Nirenberg type:

$$\|Bx\|_{E_0} \leq C \|x\|_{E_1}^\lambda \|x\|_{E_0}^{1-\lambda}$$

for any $x \in E_1$ where $\lambda \in (0, 1)$. Then using Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1\,,$$

with $p = 1/\lambda$ and $q = 1/(1 - \lambda)$. it is easy to verify that B has zero relative norm.

LEMMA 3.3. [**Paz83**, Section 2.1] *The set $Hol(E_1, E_0)$ is closed with respect to perturbations by linear operators with zero relative norm, i.e. if $A \in Hol(E_1, E_0)$ and $B \in L(E_1, E_0)$ has zero relative norm then $A + B \in Hol(E_1, E_0)$.*

Neither the theory of C^0 semigroups (c.f. Pazy [**Paz83**]) nor the theory of analytic semigroups (c.f. Henry [**Hen81**]) are able to handle fully nonlinear parabolic equations. This is mainly due to the method of integral equation which is suitable for semilinear equations only. The second reason why these methods cannot provide a local existence result is due to the fact that semigroup theories are working with function spaces which are fractional powers of the domain of a generator of an analytic semigroup (see [**Hen81**]). Therefore we need a more robust theory capable of handling fully nonlinear parabolic equations. This theory is due to Angenent and Lunardi [**Ang90a, Lun82**] and it is based on abstract results by Da Prato and Grisvard [**DPG75, DPG79**]. The basic idea is the linearization technique where one can linearize the fully nonlinear equation at the initial condition ϕ^0. Then one sets up a linearized semilinear equation with the right hand side which is of the second order with respect to deviation from the initial condition. In what follows, we shall present key steps of this method. First we need to introduce the maximal regularity class which will enable us to construct an inversion operator to a nonhomogeneous semilinear equation.

Let $E = (E_1, E_0)$ be a pair of Banach spaces for which E_1 is densely included in E_0. Let us define the following function spaces

$$X = C([0,1], E_0), \qquad Y = C([0,1], E_1) \cap C^1([0,1], E_0)\,.$$

We shall identify ∂_t with the bounded differentiation operator from Y to X defined by $(\partial_t \phi)(t) = \phi'(t)$. For a given linear bounded operator $A \in L(E_1, E_0)$ we define the extended operator $\mathcal{A} : Y \to X \times E_1$ defined by $\mathcal{A}\phi = (\partial_t \phi - A\phi, \phi(0))$. Next we define a class $\mathcal{M}_1(E)$ as follows:

$$\mathcal{M}_1(E) = \{A \in Hol(E),\ \mathcal{A} \text{ is an isomorphism between } Y \text{ and } X \times E_1\}\,.$$

It means that the class $\mathcal{M}_1(E)$ consist of all generators of analytic semigroups A such that the initial value problem for the semilinear evolution equation

$$\partial_t \phi - A\phi = f(t), \qquad \phi(0) = \phi^0,$$

has a unique solution $\phi \in Y$ for any right-hand side $f \in X$ and the initial condition $\phi^0 \in E_1$ (c.f. [**Ang90a**]). For such an operator A we obtain boundedness of the inverse of the operator $\phi \mapsto (\partial_t - A)\phi$ mapping the Banach space $Y^{(0)} = \{\phi \in Y, \phi(0) = 0\}$ onto the Banach space X, i.e.

$$\|(\partial_t - A)^{-1}\|_{L(X, Y^{(0)})} \le C < \infty\,.$$

The class $\mathcal{M}_1(E)$ is refereed to as maximal regularity class for the pair of Banach spaces $E = (E_1, E_0)$.

An analogous perturbation result to Lemma 3.3 has been proved by Angenent.

Lemma 3.4. [**Ang90a**, Lemma 2.5] *The set $\mathcal{M}_1(E_1, E_0)$ is closed with respect to perturbations by linear operators with zero relative norm.*

Using properties of the class $\mathcal{M}_1(E)$ we are able to state the main result on the local existence of a smooth solution to the abstract fully nonlinear evolutionary problem (3.3)–(3.4).

Theorem 3.5. [**Ang90a**, Theorem 2.7] *Assume that $\mathcal{F}$ is a C^1 mapping from some open subset $\mathcal{O} \subset E_1$ of the Banach space E_1 into the Banach space E_0. If the Frechét derivative $A = d\mathcal{F}(\phi)$ belongs to $\mathcal{M}_1(E)$ for any $\phi \in \mathcal{O}$ and the initial condition ϕ^0 belongs to $\mathcal{O}$ then the abstract fully nonlinear evolutionary problem (3.3)–(3.4) has a unique solution $\phi \in C^1([0,T], E_0) \cap C([0,T], E_1)$ on some small time interval $[0,T], T > 0$.*

Proof. The proof is based on the Banach fixed point theorem. Without loss of generality (by shifting the solution $\phi(t) \mapsto \phi^0 + \phi(t)$) we may assume $\phi^0 = 0$. Taylor's series expansion of $\mathcal{F}$ at $\phi = 0$ yields $\mathcal{F}(\phi) = \mathcal{F}_0 + A\phi + R(\phi)$ where

$\mathcal{F}_0 \in E_0$, $A \in \mathcal{M}_1(E)$ and the remainder function R is quadratically small, i.e. $\|R(\phi)\|_{E_0} = O(\|\phi\|^2_{E_1})$ for small $\|\phi\|_{E_1}$. Problem (3.3)–(3.4) is therefore equivalent to the fixed point problem

$$\phi = (\partial_t - A)^{-1}(R(\phi) + \mathcal{F}_0)$$

on the Banach space $Y_T^{(0)} = \{\phi \in C^1([0,T],E_0) \cap C([0,T],E_1), \phi(0) = 0\}$. Using boundedness of the operator $(\partial_t - A)^{-1}$ and taking $T > 0$ sufficiently one can prove that the right hand side of the above equation is a contraction mapping on the space $Y_T^{(0)}$ proving thus the statement of theorem. □

3.1.2.2. *Application of the abstract result for the fully nonlinear parabolic equation for the distance function.* Now we are in a position to apply the abstract result contained in Theorem 3.5 to the fully nonlinear parabolic equation (3.1) for the distance function ϕ subject to a zero initial condition $\phi^0 = 0$. Notice that one has to carefully choose function spaces to work with. Baillon in [**Bai80**] showed that, if we exclude the trivial case $E_1 = E_0$, the class $\mathcal{M}_1(E_1, E_0)$ is nonempty only if the Banach space E_0 contains a closed subspace isomorphic to the sequence space (c_0). As a consequence of this criterion we conclude that $\mathcal{M}_1(E_1, E_0)$ is empty for any reflexive Banach space E_0. Therefore the space E_0 cannot be reflexive. On the other hand, one needs to prove that the linearization $A = d\mathcal{F}(\phi) : E_1 \to E_0$ generates an analytic semigroup in E_0. Therefore it is convenient to work with little Hölder spaces satisfying these structural assumptions.

Applying the abstract result from Theorem 3.5 we are able to state the following theorem which is a special case of a more general result by Angenent [**Ang90b**, Theorem 3.1] to evolution of planar curves.

THEOREM 3.6. [**Ang90b**, Theorem 3.1] *Assume that the normal velocity $\beta = \beta(k,\nu)$ is a $C^{1,1}$ smooth function such that $\beta'_k > 0$ for all $k \in \mathbb{R}$ and $\nu \in [0, 2\pi]$. Let Γ^0 be an embedded smooth curve with Hölder continuous curvature. Then there exists a unique maximal solution $\Gamma^t, t \in [0, T_{max})$, consisting of curves evolving with the normal velocity equal to $\beta(k,\nu)$.*

Remark. Verification of nonemptyness of the set $\mathcal{M}_1(E_1, E_0)$ might be difficult for a particular choice of Banach pair (E_1, E_0). There is however a general construction of the Banach pair (E_1, E_0) such that a given linear operator A belongs to $\mathcal{M}_1(E_1, E_0)$. Let $F = (F_1, F_0)$ be a Banach pair. Assume that $A \in Hol(F_1, F_0)$. We define the Banach space $F_2 = \{\phi \in F_1, A\phi \in F_1\}$ equipped with the graph norm $\|\phi\|_{F_2} = \|\phi\|_{F_1} + \|A\phi\|_{F_1}$. For a fixed $\sigma \in (0,1)$ we introduce the continuous interpolation spaces $E_0 = F_\sigma = (F_1, F_0)_\sigma$ and $E_1 = F_{1+\sigma} = (F_2, F_1)_\sigma$. Then, by result due to Da Prato and Grisvard [**DPG75, DPG79**] we have $A \in \mathcal{M}_1(E_1, E_0)$.

3.1.3. Local existence, uniqueness and continuation of classical solutions. In this section we present another approach for the proof a of a local existence of a classical solution. Now we put our attention to a solution of the system of parabolic-ordinary differential equations (2.10) – (2.13). Let a regular smooth initial curve $\Gamma_0 = \mathrm{Img}(x_0)$ be given. Recall that a family of planar curves $\Gamma^t = \mathrm{Img}(x(.,t)),\quad t \in [0,T)$, satisfying (1.1) can be represented by a solution $x = x(u,t)$ to the position vector equation (2.4). Notice that $\beta = \beta(x,k,\nu)$ depends on x, k, ν and this is why we have to provide and analyze a closed system of equations for the variables k, ν as well as the local length $g = |\partial_u x|$ and position vector x. In the case of a nontrivial tangential velocity functional α the system of parabolic–ordinary governing equations has the following form:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta \,, \tag{3.5}$$

$$\partial_t \nu = \beta'_k \partial_s^2 \nu + (\alpha + \beta'_\nu) \partial_s \nu + \nabla_x \beta . \vec{T} \,, \tag{3.6}$$

$$\partial_t g = -g k \beta + \partial_u \alpha \,, \tag{3.7}$$

$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \tag{3.8}$$

where $(u,t) \in Q_T = [0,1] \times (0,T)$, $\mathrm{d}s = g \, \mathrm{d}u$, $\vec{T} = \partial_s x = (\cos\nu, \sin\nu)$, $\vec{N} = \vec{T}^\perp = (-\sin\nu, \, \cos\nu)$, $\beta = \beta(x,k,\nu)$. A solution (k,ν,g,x) to (3.5) – (3.8) is subject to initial conditions

$$k(.,0) = k_0 \,, \;\; \nu(.,0) = \nu_0 \,, \;\; g(.,0) = g_0 \,, \;\; x(.,0) = x_0(.)$$

and periodic boundary conditions at $u = 0,1$ except of ν for which we require the boundary condition $\nu(1,t) \equiv \nu(0,t) \bmod(2\pi)$. The initial conditions for k_0, ν_0, g_0 and x_0 have to satisfy natural compatibility constraints: $g_0 = |\partial_u x_0| > 0 \,, \; k_0 = g_0^{-3} \partial_u x_0 \wedge \partial_u^2 x_0 \,, \; \partial_u \nu_0 = g_0 k_0$ following from the equation $k = \partial_s x \wedge \partial_s^2 x$ and Frenét's formulae applied to the initial curve $\Gamma_0 = \mathrm{Img}(x_0)$. Notice that the system of governing equations consists of coupled parabolic-ordinary differential equations.

Since α enters the governing equations a solution k, ν, g, x to (3.5) – (3.8) does depend on α. On the other hand, the family of planar curves $\Gamma^t = \mathrm{Img}(x(.,t)), t \in [0,T)$, is independent of a particular choice of the tangential velocity α as it does not change the shape of a curve. The tangential velocity α can be therefore considered as a free parameter to be suitably determined later. For example, in the Euclidean curve shortening equation $\beta = k$ we can write equation (2.4) in the form $\partial_t x = \partial_s^2 x = g^{-1} \partial_u (g^{-1} \partial_u x) + \alpha g^{-1} \partial_u x$ where $g = |\partial_u x|$. Epstein and Gage [**EG87**] showed how this degenerate parabolic equation (g need not be smooth enough) can be turned into the strictly parabolic equation $\partial_t x = \partial_s^2 x = g^{-2} \partial_u^2 x)$ by choosing the tangential term α in the form $\alpha = g^{-1} \partial_u (g^{-1}) \partial_u x$. This trick is known as "De Turck's trick" named after De Turck (see [**DeT83**]) who use this approach to prove short time existence for the Ricci flow. Numerical aspects of this "trick" has been discussed by Dziuk and Deckelnick in [**Dzi94, Dzi99, Dec97**]. In general, we allow the tangential velocity functional α appearing in (3.5) – (3.8) to be dependent on k, ν, g, x in various ways including nonlocal dependence, in particular (see the next section for details).

Let us denote $\Phi = (k, \nu, g, x)$. Let $0 < \varrho < 1$ be fixed. By E_k we denote the following scale of Banach spaces (manifolds)

$$E_k = h^{2k+\varrho} \times h_*^{2k+\varrho} \times h^{1+\varrho} \times (h^{2+\varrho})^2 \tag{3.9}$$

where $k = 0, 1/2, 1$, and $h^{2k+\varrho} = h^{2k+\varrho}(S^1)$ is the "little" Hölder space (see [**Ang90a**]). By $h_*^{2k+\varrho}(S^1)$ we have denoted the Banach manifold $h_*^{2k+\varrho}(S^1) = \{\nu : \mathbb{R} \to \mathbb{R} \,, \vec{N} = (-\sin\nu, \cos\nu) \in (h^{2k+\varrho}(S^1))^2\}$. [1]

[1] Alternatively, one may consider the normal velocity β depending directly on the unit inward normal vector $\vec{N}$ belonging to the linear vector space $(h^{2k+\varrho}(S^1))^2$, i.e. $\beta = \beta(k, x, \vec{N})$.

Concerning the tangential velocity α we shall make a general regularity assumption:

$$\alpha \in C^1(\mathcal{O}_{\frac{1}{2}}, h^{2+\varrho}(S^1)) \tag{3.10}$$

for any bounded open subset $\mathcal{O}_{\frac{1}{2}} \subset E_{\frac{1}{2}}$ such that $g > 0$ for any $(k, \nu, g, x) \in \mathcal{O}_{\frac{1}{2}}$.

In the rest of this section we recall a general result on local existence and uniqueness a classical solution of the governing system of equations (3.5) – (3.8). The normal velocity β depending on k, x, ν belongs to a wide class of normal velocities for which local existence of classical solutions has been shown in [**MS04a, MS04b**]. This result is based on the abstract theory of nonlinear analytic semigroups developed by Angenent in [**Ang90a**] an it utilizes the so-called maximal regularity theory for abstract parabolic equations.

THEOREM 3.7. *(*[**MS04a**, Theorem 3.1] *Assume $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where k_0 is the curvature, ν_0 is the tangential vector, $g_0 = |\partial_u x_0| > 0$ is the local length element of an initial regular closed curve $\Gamma_0 = Img(x_0)$ and the Banach space E_k is defined as in (3.9). Assume $\beta = \beta(x, k, \nu)$ is a C^4 smooth and 2π-periodic function in the ν variable such that $\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$ and α satisfies (3.10). Then there exists a unique solution $\Phi = (k, \nu, g, x) \in C([0,T], E_1) \cap C^1([0,T], E_0)$ of the governing system of equations (3.5) – (3.8) defined on some small time interval $[0,T]$, $T > 0$. Moreover, if Φ is a maximal solution defined on $[0, T_{max})$ then we have either $T_{max} = +\infty$ or $\liminf_{t\to T_{max}^-} \min_{\Gamma^t} \beta'_k(x,k,\nu) = 0$ or $T_{max} < +\infty$ and $\max_{\Gamma^t} |k| \to \infty$ as $t \to T_{max}$.*

PROOF. Since $\partial_s \nu = k$ and $\partial_s \beta = \beta'_k \partial_s k + \beta'_\nu k + \nabla_x \beta . \vec{T}$ the curvature equation (3.5) can be rewritten in the divergent form

$$\partial_t k = \partial_s(\beta'_k \partial_s k) + \partial_s(\beta'_\nu k) + k \nabla_x \beta . \vec{N} + \partial_s(\nabla_x \beta . \vec{T}) + \alpha \partial_s k + k^2 \beta \,.$$

Let us take an open bounded subset $\mathcal{O}_{\frac{1}{2}} \subset E_{\frac{1}{2}}$ such that $\mathcal{O}_1 = \mathcal{O}_{\frac{1}{2}} \cap E_1$ is an open subset of E_1 and $\Phi_0 \in \mathcal{O}_1$, $g > 0$, and $\beta'_k(x, k, \nu) > 0$ for any $(k, \nu, g, x) \in \mathcal{O}_1$. The linearization of f at a point $\bar\Phi = (\bar k, \bar\nu, \bar g, \bar x) \in \mathcal{O}_1$ has the form $df(\bar\Phi) = d_\Phi F(\bar\Phi, \bar\alpha) + d_\alpha F(\bar\Phi, \bar\alpha)\, d_\Phi \alpha(\bar\Phi)$ where $\bar\alpha = \alpha(\bar\Phi)$ and

$$d_\Phi F(\bar\Phi, \bar\alpha) = \partial_u \bar D \partial_u + \bar B \partial_u + \bar C \,, \quad d_\alpha F(\bar\Phi, \bar\alpha) = \left(\bar g^{-1} \partial_u \bar k \,,\ \bar k \,,\ \partial_u \,,\ \bar{\vec{T}}\right)$$

$\bar D = \mathrm{diag}(\bar D_{11}, \bar D_{22}, 0, 0, 0)$, $\bar D_{11} = \bar D_{22} = \bar g^{-2} \beta'_k(\bar x, \bar k, \bar\nu) \in C^{1+\varrho}(S^1)$ and $\bar B, \bar C$ are 5×5 matrices with $C^\varrho(S^1)$ smooth coefficients. Moreover, $\bar B_{ij} = 0$ for $i = 3, 4, 5$ and $\bar C_{3j} \in C^{1+\varrho}$, $\bar C_{ij} \in C^{2+\varrho}$ for $i = 4, 5$ and all j. The linear operator A_1 defined by $A_1 \Phi = \partial_u(\bar D \partial_u \Phi)$, $D(A_1) = E_1 \subset E_0$ is a generator of an analytic semigroup on E_0 and, moreover, $A_1 \in \mathcal{M}_1(E_0, E_1)$ (see [**Ang90a, Ang90b**]). Notice that $d_\alpha F(\bar\Phi, \bar\alpha)$ belongs to $\mathcal{L}(C^{2+\varrho}(S^1), E_{\frac{1}{2}})$ and this is why we can write $d_\Phi f(\bar\Phi)$ as a sum $A_1 + A_2$ where $A_2 \in L(E_{\frac{1}{2}}, E_0)$. By Gagliardo–Nirenberg inequality we have $\|A_2 \Phi\|_{E_0} \le C \|\Phi\|_{E_{\frac{1}{2}}} \le C \|\Phi\|_{E_0}^{1/2} \|\Phi\|_{E_1}^{1/2}$ and so the linear operator A_2 is a relatively bounded linear perturbation of A_1 with zero relative bound (cf. [**Ang90a**]). With regard to Lemma 3.4 (see also [**Ang90a**, Lemma 2.5]) the class $\mathcal{M}_1$ is closed with respect to such perturbations. Thus $d_\Phi f(\bar\Phi) \in \mathcal{M}_1(E_0, E_1)$. The proof of the short time existence of a solution Φ now follows from Theorem 3.5 (see also [**Ang90a**, Theorem 2.7]).

Finally, we will show that the maximal curvature becomes unbounded as $t \to T_{max}$ in the case $\liminf_{t\to T_{max}^-} \min_{\Gamma^t} \beta'_k > 0$ and $T_{max} < +\infty$. Suppose to the contrary that $\max_{\Gamma^t} |k| \le M < \infty$ for any $t \in [0, T_{max})$. According to [**Ang90b**, Theorem 3.1] there exists a unique maximal solution $\Gamma : [0, T'_{max}) \to \Omega(\mathbb{R}^2)$ satisfying the geometric equation (1.1). Recall that $\Omega(\mathbb{R}^2)$ is the space of C^1 regular Jordan curves in the plane (cf. [**Ang90b**]). Moreover, Γ^t is a C^∞ smooth curve for any $t \in (0, T'_{max})$ and the maximum of the absolute value of the curvature tends to infinity as $t \to T'_{max}$. Thus $T_{max} < T'_{max}$ and therefore the curvature and subsequently ν remain bounded in $C^{2+\varrho'}$ norm on the interval $[0, T_{max}]$ for any $\varrho' \in (\varrho, 1)$. Applying the compactness argument one sees that the limit $\lim_{t\to T_{max}} \Phi(., t)$ exists and remains bounded in the space E_1 and one can continue the solution Φ beyond T_{max}, a contradiction. □

Remark. In a general case where the normal velocity may depend on the position vector x, the maximal time of existence of a solution can be either finite or infinite. Indeed, as an example one can consider the unit ball $B = \{|x| < 1\}$ and function $\delta(x) = (|x| - 1)^\gamma$ for $x \notin B$, $\gamma > 0$. Suppose that $\Gamma_0 = \{|x| = R_0\}$ is a circle with a radius $R_0 > 1$ and the family $\Gamma^t, t \in [0, T)$, evolves according to the normal velocity function $\beta(x, k) = \delta(x)k$. Then, it is an easy calculus to verify that the family Γ^t approaches the boundary $\partial B = \{|x| = 1\}$ in a finite time $T_{max} < \infty$ provided that $0 < \gamma < 1$ whereas $T_{max} = +\infty$ in the case $\gamma = 1$.

CHAPTER 4

Level set methods for curvature driven flows of planar curves

By contrast to the direct approach, *level set methods* are based on introducing an auxiliary shape function whose zero level sets represent a family of planar curves which is evolved according to the geometric equation (1.1) (see e.g. [**OS88, Set90, Set96, Set98**]). The level set approach handles implicitly the curvature-driven motion, passing the problem to higher dimensional space. One can deal with splitting and/or merging of evolving curves in a robust way. However, from the computational point of view, level set methods are much more computationaly expensive than methods based on the direct approach. The purpose of this chapter is to present basic ideas and results concerning the level set approach in curvature driven flows of planar curves.

Other indirect method is based on the phase-field formulations. In these lecture notes we however do not go into details of these methods and interested reader is referred to extensive literature in this topic (see e.g. [**Cag90, EPS96, BM98**] and references therein).

4.1. Level set representation of Jordan curves in the plane

In the level set method the evolving family of planar curves $\Gamma^t, t \geq 0$, is represented by the zero level set of the so-called shape function $\phi : \Omega \times [0, T] \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^2$ is a simply connected domain containing the whole family of evolving curves $\Gamma^t, t \in [0, T]$. We adopt a notation according to which the interior of a curve is described as: $int(\Gamma^t) = \{x \in \mathbb{R}^2, \phi(x, t) < 0\}$ and, consequently,

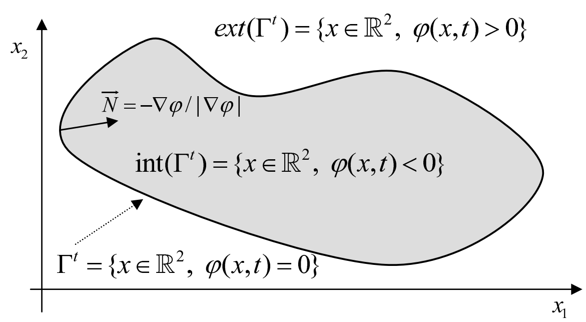

FIGURE 1. Description of the level set representation of a planar embedded curve by a shape function $\phi : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$.

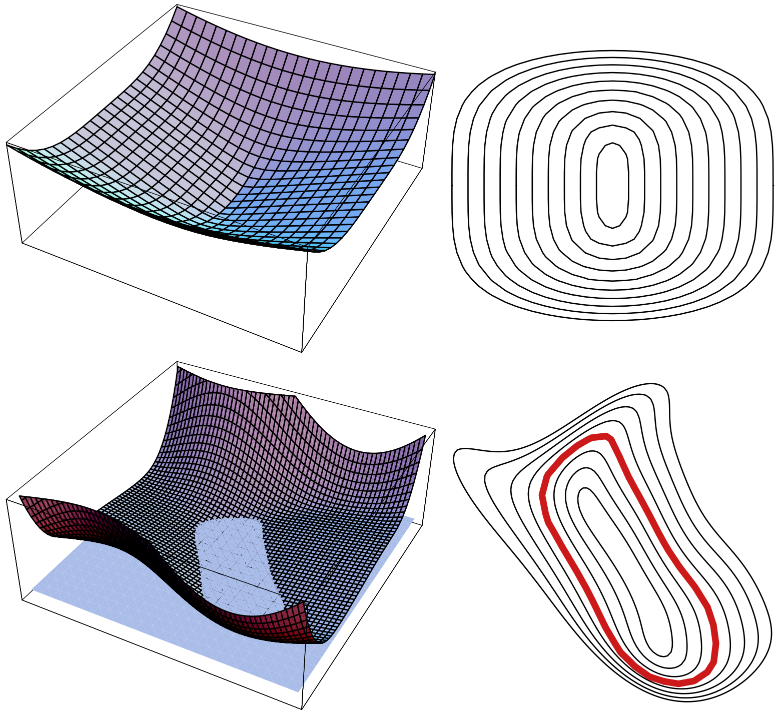

FIGURE 2. Description of the representation of planar embedded curves by level sets of two functions $\phi : \mathbb{R}^2 \to \mathbb{R}$. The level set functions (left) and their level cross-section (right).

$ext(\Gamma^t) = \{x \in \mathbb{R}^2, \phi(x,t) > 0\}$ and $\Gamma^t = \{x \in \mathbb{R}^2, \phi(x,t) = 0\}$ (see Fig. 1). With this convection, the unit inward normal vector $\vec{N}$ can be expressed as

$$\vec{N} = -\nabla\phi/|\nabla\phi|\,.$$

In order to express the signed curvature k of the curve Γ^t we make use of the identity $\phi(x(s,t),t) = 0$. Differentiating this identity with respect to the arc-length parameter s we obtain $0 = \nabla\phi.\partial_s x = \nabla\phi.\vec{T}$. Differentiating the latter identity with respect to s again and using the Frenét formula $\partial_s \vec{T} = k\vec{N}$ we obtain $0 = k(\nabla\phi.\vec{N}) + \vec{T}^\perp \nabla^2\phi\vec{T}$. Since $\vec{N} = -\nabla\phi/|\nabla\phi|$ we have

$$k = \frac{1}{|\nabla\phi|}\vec{T}^T\nabla^2\phi\vec{T}\,. \tag{4.1}$$

It is a long but straightforward computation to verify the identity

$$|\nabla\phi|\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = \vec{T}^\perp\nabla^2\phi\vec{T}\,.$$

Hence the signed curvature k is given by the formula

$$k = \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\,.$$

In other words, the curvature k is just the minus the divergence of the normal vector $\vec{N} = \nabla\phi/|\nabla\phi|$, i.e. $k = -\mathrm{div}\vec{N}$.

Let us differentiate the equation $\phi(x(s,t),t) = 0$ with respect to time. We obtain $\partial_t\phi + \nabla\phi.\partial_t x = 0$. Since the normal velocity of x is $\beta = \partial_t x.\vec{N}$ and $\vec{N} = -\nabla\phi/|\nabla\phi|$ we obtain

$$\partial_t\phi = |\nabla\phi|\beta\,.$$

Combining the above identities for $\partial_t\phi, \vec{N}$, and k we conclude that the geometric equation (1.1) can be reformulated in terms of the evolution of the shape function $\phi = \phi(x,t)$ satisfying the following fully nonlinear parabolic equation:

$$\partial_t\phi = |\nabla\phi|\beta\left(\mathrm{div}\left(\nabla\phi/|\nabla\phi|\right), x, -\nabla\phi/|\nabla\phi|\right), \qquad x \in \Omega,\ t \in (0,T)\,. \tag{4.2}$$

Here we assume that the normal velocity β may depend on the curvature k, the position vector x and the tangent angle ν expressed through the unit inward normal vector $\vec{N}$, i.e. $\beta = \beta(k, x, \vec{N})$. Since the behavior of the shape function ϕ in a far distance from the set of evolving curves $\Gamma^t, t \in [0,T]$, does not influence their evolution, it is usual in the context of the level set equation to prescribe homogeneous Neumann boundary conditions at the boundary $\partial\Omega$ of the computational domain Ω, i.e.

$$\phi(x,t) = 0 \quad \text{for } x \in \partial\Omega\,. \tag{4.3}$$

The initial condition for the level set shape function ϕ can be constructed as a signed distance function measuring the signed distance of a point $x \in \mathbb{R}^2$ and the initial curve Γ^0, i.e.

$$\phi(x,0) = \mathrm{dist}(x,\Gamma^0) \tag{4.4}$$

where $\mathrm{dist}(x,\Gamma^0)$ is a signed distance function defined as

$$\begin{aligned}
\mathrm{dist}(x,\Gamma^0) &= \inf_{y\in\Gamma^0} |x-y|, \qquad \text{for } x \in ext(\Gamma^0)\,,\\
\mathrm{dist}(x,\Gamma^0) &= -\inf_{y\in\Gamma^0} |x-y|, \qquad \text{for } x \in int(\Gamma^0)\,,\\
\mathrm{dist}(x,\Gamma^0) &= 0, \qquad \text{for } x \in \Gamma^0\,.
\end{aligned}$$

If we assume that the normal velocity of an evolving curve Γ^t is an affine in the k variable, i.e.

$$\beta = \mu k + f$$

where $\mu = \mu(x,\vec{N})$ is a coefficient describing dependence of the velocity speed on the position vector x and the orientation of the curve Γ^t expressed through the unit inward normal vector $\vec{N}$ and $f = f(x,\vec{N})$ is an external forcing term.

$$\partial_t\phi = \mu\,|\nabla\phi|\,\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + f|\nabla\phi|, \qquad x \in \Omega, t \in (0,T)\,. \tag{4.5}$$

4.1.1. A-priori bounds for the total variation of the shape function. In this section we derive an important a-priori bound for the total variation of the shape function satisfying the level set equation (4.2). The total variation (or the $W^{1,1}$ Sobolev norm) of the function $\phi(.,t)$ is defined as $\int_\Omega |\nabla\phi(x,t)|\,\mathrm{d}x$ where

$\Omega \subset \mathbb{R}^2$ is a simply connected domain such that $int(\Gamma^t) \subset \Omega$ for any $t \in [0, T]$. Differentiating the total variation of $\phi(., t)$ with respect to time we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \phi| \,\mathrm{d}x &= \int_\Omega \frac{1}{|\nabla \phi|} (\nabla \phi . \partial_t \nabla \phi) \,\mathrm{d}x = \int_\Omega \frac{\nabla \phi}{|\nabla \phi|} . \nabla \partial_t \phi \,\mathrm{d}x \\ &= -\int_\Omega \mathrm{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) . \partial_t \phi \,\mathrm{d}x = -\int_\Omega k\beta |\nabla \phi| \,\mathrm{d}x \end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \phi| \,\mathrm{d}x + \int_\Omega k\beta |\nabla \phi| \,\mathrm{d}x = 0 \tag{4.6}$$

where k is expressed as in (4.1) and $\beta = \beta\left(\mathrm{div}\left(\nabla\phi/|\nabla\phi|\right), x, -\nabla\phi/|\nabla\phi|\right)$. With help of the co-area integration theorem, the identity (4.6) can be viewed as a level set analogy to the total length equation (2.14).

In the case of the Euclidean curvature driven flow when curves are evolved in the normal direction by the curvature (i.e. $\beta = k$) we have $\int_\Omega k\beta |\nabla \phi| \,\mathrm{d}x = \int_\Omega k^2 |\nabla \phi| \,\mathrm{d}x > 0$ and this is why

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \phi| \,\mathrm{d}x < 0 \quad \text{for any } t \in (0, T),$$

implying thus the estimate

$$\phi \in L^\infty((0, T), W^{1,1}(\Omega)). \tag{4.7}$$

The same property can be easily proved by using Gronwall's lemma for a more general form of the normal velocity when $\beta = \mu k + f$ where $\mu = \mu(x, \vec{N}) > 0, f = f(x, \vec{N})$ are globally bounded functions. We presented this estimate because the same estimates can be proved for the gradient flow in the theory of minimal surfaces. Notice that the estimate (4.7) is weaker than the L^2–energy estimate $\phi \in L^\infty((0, T), W^{1,2}(\Omega))$ which can be easily shown for nondegenerate parabolic equation of the form $\partial_t \phi = \Delta \phi$, $\mathrm{d}\phi/\mathrm{d}n = 0$ on $\partial\Omega$, by multiplying the equation with the test function ϕ and integrating over the domain Ω.

4.2. Viscosity solutions to the level set equation

In this section we briefly describe a concept of viscosity solutions to the level set equation (4.2). We follow the book by Cao (c.f. [**Cao03**]). For the sake of simplicity of notation we shall consider the normal velocity β of the form $\beta = \beta(k)$. Hence equation (4.2) has a simplified form

$$\partial_t \phi = |\nabla \phi| \beta \left(\mathrm{div}\left(\nabla \phi / |\nabla \phi|\right)\right). \tag{4.8}$$

The concept of viscosity solutions has been introduced by Crandall and Lions in [**CL83**]. It has been generalized to second order PDEs by Jensen [**Jen88**] (see also [**IS95, FS93**]). The proof of the existence and uniqueness of a viscosity solution to (4.8) is a consequence of the maximum principle for viscosity solutions (uniqueness part). Existence part can be proven by the method of sub and supersolutions known as the so-called Perron's method.

Following [**Cao03**] we first explain the basic idea behind the definition of a viscosity solution. We begin with a simple linear parabolic equation

$$\partial_t \phi = \Delta \phi \,. \tag{4.9}$$

Let ψ be any C^2 smooth function such that $\phi - \psi < 0$ except of some point $(\bar{x}, \bar{t})$ in which $\phi(\bar{x}, \bar{t}) = \psi(\bar{x}, \bar{t})$, i.e. $(\bar{x}, \bar{t})$ is a strict local maximum of the function $\phi - \psi$. Clearly, $\nabla\phi(\bar{x}, \bar{t}) - \nabla\psi(\bar{x}, \bar{t}) = 0, \partial_t\phi(\bar{x}, \bar{t}) - \partial_t\psi(\bar{x}, \bar{t}) = 0$, and $\Delta(\phi(\bar{x}, \bar{t}) - \psi(\bar{x}, \bar{t})) \leq 0$. Hence

$$\partial_t \psi \leq \Delta \psi \quad \text{at} \quad (\bar{x}, \bar{t}) \,. \tag{4.10}$$

We say that ϕ is a subsolution to (4.9) if the inequality (4.10) hold whenever $\phi - \psi$ has a strict maximum at $(\bar{x}, \bar{t})$. Analogously, we say that ϕ is a supersolution to (4.9) if the reverse inequality $\partial_t \psi \geq \Delta \psi$ holds at a point $(\bar{x}, \bar{t})$ in which the function $\phi - \psi$ attains a strict minimum. It is important to realize, that such a definition of a sub and supersolution does not explicitly require smoothness of the function ϕ. It has been introduced by Crandall and Lions in [**CL83**]. Moreover, the above concept of sub and supersolutions can be extended to the case when the second order differential operator contains discontinuities. For the Euclidean motion by mean curvature (i.e. $\beta(k) = k$) the existence and uniqueness of a viscosity solution to (4.8) has been established by Evans and Spruck [**ES91**] and by Chen, Giga and Goto [**CGG91**] for the case $\beta(k)$ is sublinear at $\pm\infty$. Finally, Barles, Souganidis and Ishii introduced a concept of a viscosity solution for (4.8) in the case of arbitrary continuous and nondecreasing function $\beta(k)$ and they also proved the existence and uniqueness of a viscosity solution in [**IS95, BS91**]. Moreover, Souganidis extended a notion of a viscosity solution for the case when the elliptic operator is undefined in a set of critical points of ϕ.

Following Souganidis et al. (c.f. [**IS95, BS91**]), the class $\mathcal{A}(\beta)$ of admissible test functions consists of those C^2 compactly supported functions $\psi : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ having the property: if $(\bar{x}, \bar{t})$ is a critical point of ψ, i.e. $\nabla\psi(\bar{x}, \bar{t}) = 0$ then there exists a neighborhood $B_\delta(\bar{x}, \bar{t})$ with a radius $\delta > 0$, a function $f \in \mathcal{F}(\beta)$, and $\omega \in C((0, \infty))$ satisfying $\lim_{r\to 0} \omega(r)/r = 0$ such that

$$|\psi(y, s) - \psi(\bar{x}, \bar{t}) - \partial_t\psi(\bar{x}, \bar{t})(s - \bar{t})| \leq f(|y - \bar{x}|) + \omega(|s - \bar{t}|), \quad \text{for any } (y, s) \in B_\delta(\bar{x}, \bar{t}) \,.$$

The class $\mathcal{F}(\beta)$ consists of those C^2 functions f such that $f(0) = f'(0) = f''(0) = 0, f''(r) > 0$ for $r > 0$ and $\lim_{r\to 0} f'(|r|)\beta(1/r) = 0$.

The idea behind a relatively complicated definition of the set of admissible function is simple. It consists in the requirement that test functions must be enough flat to absorb singularities of the function β at their critical points. With this concept of the set of admissible test functions we are in a position to introduce a notion of a viscosity sub and super solution to the level set equation (4.8).

Definition 4.1. [**Cao03**, Definition 4.3.2] We say that a bounded function $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a viscosity subsolution to (4.8) if for all admissible functions $\psi \in \mathcal{A}(\beta)$, if $\phi^* - \psi$ admits a strict maximum at a point $(\bar{x}, \bar{t})$ then

$$\begin{aligned} \partial_t\psi(\bar{x}, \bar{t}) &\leq |\nabla\psi(\bar{x}, \bar{t})|\beta\left(\operatorname{div}\left(\nabla\psi(\bar{x}, \bar{t})/|\nabla\psi(\bar{x}, \bar{t})|\right)\right), \qquad \text{if } \nabla\psi(\bar{x}, \bar{t}) \neq 0, \\ \partial_t\psi(\bar{x}, \bar{t}) &\leq 0, \qquad \text{if } \nabla\psi(\bar{x}, \bar{t}) = 0 \,. \end{aligned}$$

We say that a bounded function $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a viscosity supersolution to (4.8) if for all admissible functions $\psi \in \mathcal{A}(\beta)$, if $\phi_* - \psi$ admits a strict minimum at a point $(\bar{x}, \bar{t})$ then

$$\begin{array}{rcl} \partial_t \psi(\bar{x},\bar{t}) & \geq & |\nabla\psi(\bar{x},\bar{t})|\beta\left(\operatorname{div}\left(\nabla\psi(\bar{x},\bar{t})/|\nabla\psi(\bar{x},\bar{t})|\right)\right), \qquad \text{if } \nabla\psi(\bar{x},\bar{t}) \neq 0, \\ \partial_t \psi(\bar{x},\bar{t}) & \geq & 0, \qquad \text{if } \nabla\psi(\bar{x},\bar{t}) = 0\,. \end{array}$$

We say that ϕ is a viscosity solution if it both viscosity sub and supersolution.

Here we have denoted by ϕ^* and ϕ_* the upper and lower semicontinuous envelope of the function ϕ, i.e. $\phi^*(x,t) = \limsup_{(y,s)\to(x,t)} \phi(y,s)$ and $\phi_*(x,t) = \liminf_{(y,s)\to(x,t)} \phi(y,s)$.

THEOREM 4.2. [**IS95**],[**Cao03**, Theorem 4.3.2] *Let $\phi^0 \in BUC(\mathbb{R}^2)$. Assume the function $\beta : \mathbb{R} \to \mathbb{R}$ is nondecreasing and continuous. Then there exists a unique viscosity solution $\phi = \phi(x,t)$ to*

$$\begin{array}{rcl} \partial_t \phi & = & |\nabla\phi|\beta\left(\operatorname{div}\left(\nabla\phi/|\nabla\phi|\right)\right), \qquad x \in \mathbb{R}^2, t \in (0,T) \\ \phi(x,0) & = & \phi^0(x), \quad x \in \mathbb{R}^2 \end{array}$$

PROOF. The proof of this theorem is rather complicated and relies on several results from the theory of viscosity solutions. The hardest part is the proof of the uniqueness of a viscosity solution. It is based on the comparison (maximum) principle (see e.g. [**Cao03**, Theorem 4.3.1]) for viscosity sub and supersolutions to (4.8). It uses a clever result in this field which referred to as the Theorem on Sums proved by Ishii (see [**Cao03**, Lemma 4.3.1] for details). The proof of existence is again due to Ishii and is based on the Perron method of sub and supersolutions. First one has to prove that, for a set S of uniformly bounded viscosity subsolutions to (4.8), their supremum

$$\bar{\psi}(x,t) = \sup\{\psi(x,t),\ \psi \in S\}$$

is also a viscosity subsolution. If there are bounded viscosity sub and supersolutions $\underline{\psi}, \bar{\psi}$ to (4.8) such that $\underline{\psi} \leq \bar{\psi}$ then it can be shown that

$$\phi(x,t) = \sup\{\psi(x,t), \psi \text{ is a viscosity subsolution}, \underline{\psi} \leq \psi \leq \bar{\psi}\}$$

is a viscosity solution to (4.8) (c.f. [**Cao03**, Propositions 4.3.3, 4.3.4]). Finally, one has to construct suitable viscosity sub and supersolutions $\underline{\psi}, \bar{\psi}$ satisfying $\underline{\psi} \leq \phi^0 \leq \bar{\psi}$ for an initial condition ϕ^0 belonging to the space BUC of all bounded uniformly continuous functions in $\mathbb{R}^2$. The statement of the Theorem then follows. □

4.3. Numerical methods

Although these lecture notes are not particularly concerned with numerical methods for level set methods we present results obtained by a comprehensive Matlab toolbox ToolboxLS-1.1 which can be used for numerical approximation of level set methods in two or three spatial dimensions. It has been developed by Ian Mitchell and its latest version can be freely downloaded from his web page `www.cs.ubc.ca/~mitchell`.

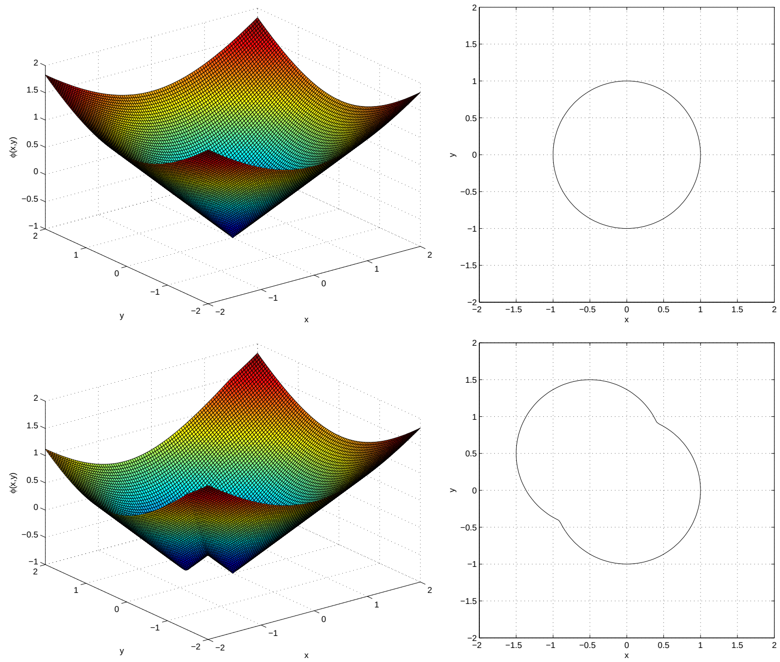

FIGURE 3. Two examples of level set functions $\phi(.,t)$ (left) and their zero level set (right) plotted at some positive time $t > 0$.

4.3.1. Examples from Mitchell's Level set Matlab toolbox. In Fig. 3 we present an output of Mitchell's ToolboxLS-1.0 for two different level set function evolution (left) for some time $t > 0$. On the right side we can see corresponding zero level sets.

The Matlab toolbox can be used for tracking evolution of two dimensional embedded surfaces in $\mathbb{R}^3$. In Fig. 4 we present evolution of a two dimensional dumb-bell like surface which is evolved by the mean curvature. Since the mean curvature for a two dimensional surface is a sum of two principal cross-sectional curvatures one can conclude that the mean curvature at the bottle-neck of the surface is positive because of the dominating principal curvature of the section plane perpendicular to the axis of a rotational symmetry of the dumb-bell. Thus the flow of a surface tends to shrink the bottle-neck. Notice that this is purely three dimensional feature and can not be observed in two dimensions. Furthermore, we can see from Fig. 4 that dumb-bell's bottle-neck shrinks to a pinching point in a finite time. After that time evolution continues in two separate sphere–like surfaces which shrink to two points in finite time. This observation enables us to conclude that a three dimensional generalization of Grayson's theorem (see Section 2) is false.

Another intuitive explanation for the failure of the Grayson theorem in three dimensions comes from the description of the mean curvature flow of two dimensional embedded surfaces in $\mathbb{R}^3$. According to Huisken [**Hui90**] the mean curvature H of

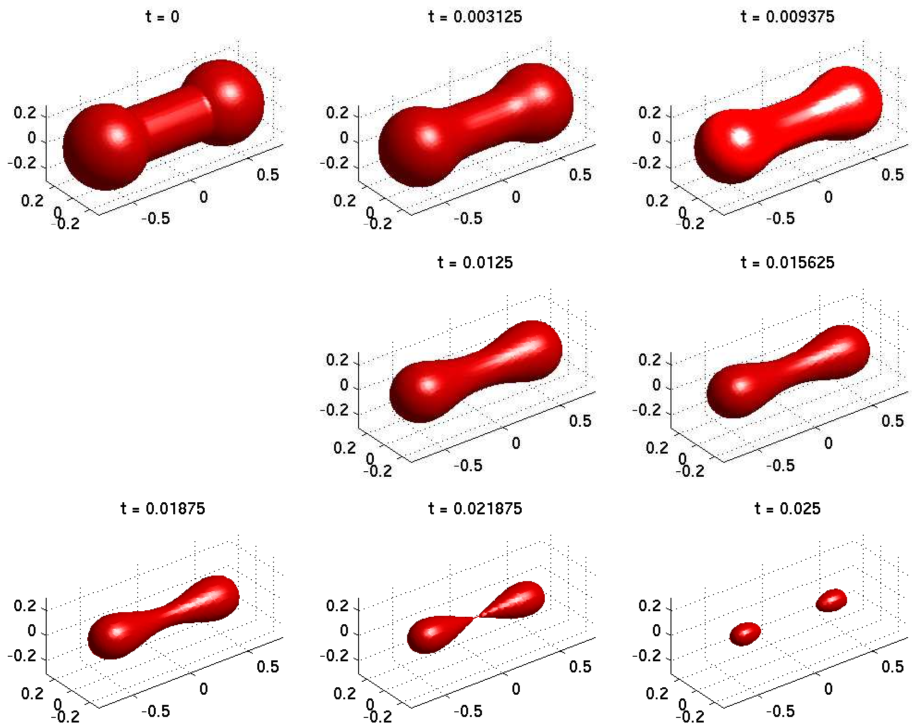

FIGURE 4. Time evolution of a dumb-bell initial surface driven by the mean curvature.

the surface is a solution to the following system of nonlinear parabolic equations

$$
\begin{aligned}
\partial_t H &= \Delta_{\mathcal{M}} H + |A|^2 H\,, \\
\partial_t |A|^2 &= \Delta_{\mathcal{M}} |A|^2 - 2|\nabla_{\mathcal{M}} A|^2 + 2|A|^4
\end{aligned}
$$

where $|A|^2$ is the second trace (Frobenius norm) of the second fundamental form of the embedded manifold $\mathcal{M}$. Here $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator with respect to the surface $\mathcal{M}$. The above system of equations is a two dimensional generalization of the simple one dimensional parabolic equation $\partial_t k = \partial_s^2 k + k^3$ describing the Euclidean flow of planar curves evolved by the curvature. Now, one can interpret Grayson's theorem for embedded curves in terms of nonincrease of nodal points of the curvature k. This result is known in the case of a scalar reaction diffusion equation and is refererred to as Sturm's theorem or Nonincrease of lap number theorem due to Matano. However, in the case of a system of two dimensional equations for the mean curvature H and the second trace $|A|^2$ one cannot expect similar result which is known to be an intrinsic property of scalar parabolic equations and cannot be extended for systems of parabolic equations.

CHAPTER 5

Numerical methods for the direct approach

In this part we suggest a fully discrete numerical scheme for the direction approach for solving the geometric equation (1.1). It is based on numerical approximation of a solution to the system of governing equations (2.10)–(2.13). The numerical scheme is semi-implicit in time, i.e. all nonlinearities are treated from the previous time step and linear terms are discretized at the current time level. Then we solve tridiagonal systems in every time step in a fast and simple way. We emphasize the role of tangential redistribution. The direct approach for solving (1.1) can be accompanied by a suitable choice of a tangential velocity α significantly improving and stabilizing numerical computations as it was documented by many authors (see e.g. [**Dec97, HLS94, HKS98, MS99, MS01, MS04a, MS04b**]). We show that stability constraint for our semi-implicit scheme with tangential redistribution is related to an integral average of $k\beta$ along the curve and not to pointwise values of $k\beta$. The pointwise influence of this term would lead to severe time step restriction in a neighborhood of corners while our approach benefits from an overall smoothness of the curve. Thus the method allows the choosing of larger time steps without loss of stability.

We remind ourselves that other popular techniques, like e.g. level-set method due to Osher and Sethian [**Set96, OF03**] or phase-field approximations (see e.g. Caginalp, Elliott et al. or Beneš [**Cag90, EPS96, Ben01, BM98**]) treat the geometric equation (1.1) by means of a solution to a higher dimensional parabolic problem. In comparison to these methods, in the direct approach one space dimensional evolutionary problems are solved only.

5.1. A role of the choice of a suitable tangential velocity

The main purpose of this section is to discuss various possible choices of a tangential velocity functional α appearing in the system of governing equations (2.10)–(2.13). In this system α can be viewed still as a free parameter which has to be determined in an appropriate way. Recall that k, ν, g, x do depend on α but the family $\Gamma_t = \mathrm{Img}(x(., t)), t \in [0, T)$, itself is independent of a particular choice of α.

To motivate further discussion, we recall some of computational examples in which the usual choice $\alpha = 0$ fails and may lead to serious numerical instabilities like e.g. formation of so-called swallow tails. In Figures 1 and 2 we computed the mean curvature flow of two initial curves (bold faced curves). We chose $\alpha = 0$ in the experiment shown in Fig. 1. It should be obvious that numerically computed grid points merge in some parts of the curve Γ_t preventing thus numerical approximation of $\Gamma_t, t \in [0, T)$, to be continued beyond some time T which is still far away from the maximal time of existence T_{max}. These examples also showed that a suitable

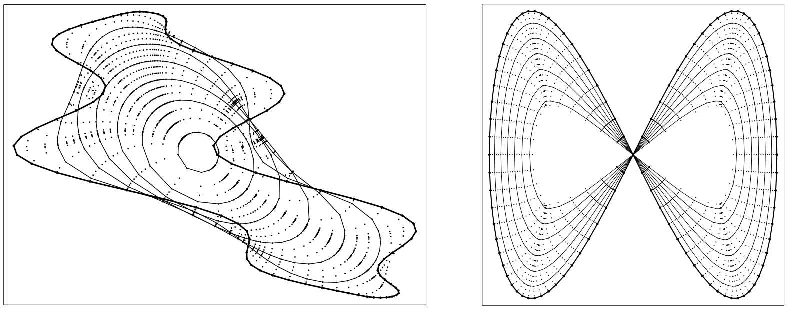

FIGURE 1. Merging of numerically computed grid points in the case of the vanishing tangential velocity functional $\alpha = 0$.

grid points redistribution governed by a nontrivial tangential velocity functional α is needed in order to compute the solution on its maximal time of existence.

The idea behind construction of a suitable tangential velocity functional α is rather simple and consists in the analysis of the quantity θ defined as follows:

$$\theta = \ln(g/L)$$

where $g = |\partial_u x|$ is a local length and L is a total length of a curve $\Gamma = \mathrm{Img}(x)$. The quantity θ can be viewed as the logarithm of the relative local length g/L. Taking into account equations (2.12) and (2.14) we have

$$\partial_t \theta + k\beta - \langle k\beta \rangle_\Gamma = \partial_s \alpha \,. \tag{5.1}$$

By an appropriate choice of $\partial_s \alpha$ in the right hand side of (5.1) appropriately we can therefore control behavior of θ. Equation (5.1) can be also viewed as a kind of a constitutive relation determining redistribution of grid point along a curve.

5.1.1. Non-locally dependent tangential velocity functional. We first analyze the case when $\partial_s \alpha$ (and so does α) depends on other geometric quantities k, β and g in a nonlocal way. The simplest possible choice of $\partial_s \alpha$ is:

$$\partial_s \alpha = k\beta - \langle k\beta \rangle_\Gamma \tag{5.2}$$

yielding $\partial_t \theta = 0$ in (5.1). Consequently,

$$\frac{g(u,t)}{L_t} = \frac{g(u,0)}{L_0} \quad \text{for any } u \in S^1,\ t \in [0, T_{max})\,.$$

Notice that α can be uniquely computed from (5.2) under the additional renormalization constraint: $\alpha(0,t) = 0$. In the sequel, tangential redistribution driven by a solution α to (5.2) will be refereed to as *a parameterization preserving relative local length*. It has been first discovered and utilized by Hou et al. in [**HLS94, HKS98**] and independently by Mikula and Ševčovič in [**MS99, MS01, MS04a, MS04b**].

A general choice of α is based on the following setup:

$$\partial_s \alpha = k\beta - \langle k\beta \rangle_\Gamma + \left(e^{-\theta} - 1\right) \omega(t) \tag{5.3}$$

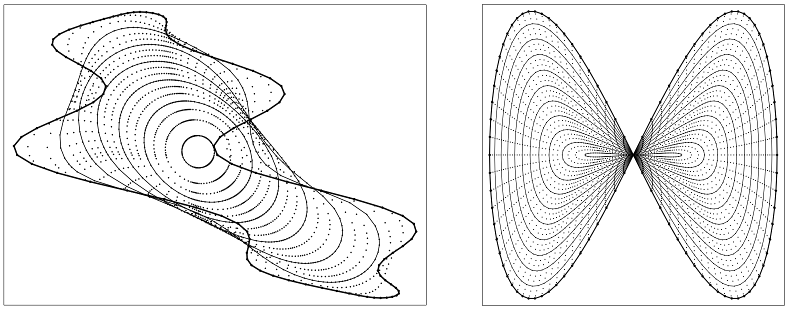

FIGURE 2. Impact of suitably chosen tangential velocity functional α on enhancement of spatial grids redistribution.

where $\omega \in L^1_{loc}([0,T_{max}))$. If we additionally suppose

$$\int_0^{T_{max}} \omega(\tau)\, d\tau = +\infty \tag{5.4}$$

then, after insertion of (5.3) into (5.1) and solving the ODE $\partial_t \theta = \left(e^{-\theta} - 1\right)\omega(t)$, we obtain $\theta(u,t) \to 0$ as $t \to T_{max}$ and hence

$$\frac{g(u,t)}{L_t} \to 1 \quad \text{as } t \to T_{max} \quad \text{uniformly w.r. to } u \in S^1.$$

In this case redistribution of grid points along a curve becomes uniform as t approaches the maximal time of existence T_{max}. We will refer to the parameterization based on (5.3) to as *an asymptotically uniform parameterization.* The impact of a tangential velocity functional defined as in (5.2) on enhancement of redistribution of grid points can be observed from two examples shown in Fig. 2 computed by Mikula and Ševčovič in [**MS01**].

Asymptotically uniform redistribution of grid points is of a particular interest in the case when the family $\{\Gamma_t, t \in [0,T)\}$ shrinks to a point as $t \to T_{max}$, i.e. $\lim_{t\to T_{max}} L_t = 0$. Then one can choose $\omega(t) = \kappa_2 \langle k\beta\rangle_{\Gamma_t}$ where $\kappa_2 > 0$ is a positive constant. By (2.14), $\int_0^t \omega(\tau)\, d\tau = -\kappa_2 \int_0^t \ln L_\tau d\tau = \kappa_2(\ln L_0 - \ln L_t) \to +\infty$ as $t \to T_{max}$. On the other hand, if the length L_t is away from zero and $T_{max} = +\infty$ one can choose $\omega(t) = \kappa_1$, where $\kappa_1 > 0$ is a positive constant in order to meet the assumption (5.4).

Summarizing, in both types of grid points redistributions discussed above, a suitable choice of the tangential velocity functional α is given by a solution to

$$\partial_s \alpha = k\beta - \langle k\beta\rangle_\Gamma + (L/g - 1)\,\omega\,,\ \alpha(0) = 0\,, \tag{5.5}$$

where $\omega = \kappa_1 + \kappa_2\langle k\beta\rangle_\Gamma$ and $\kappa_1, \kappa_2 \geq 0$ are given constants.

If we insert tangential velocity functional α computed from (5.5) into (2.10)–(2.13) and make use of the identity $\alpha\partial_s k = \partial_s(\alpha k) - k\partial_s\alpha$ then the system of governing equations can be rewritten as follows:

$$\partial_t k = \partial_s^2 \beta + \partial_s(\alpha k) + k\langle k\beta\rangle_\Gamma + (1 - L/g)\, k\omega\,, \tag{5.6}$$

$$\partial_t \nu = \beta_k' \partial_s^2 \nu + (\alpha + \beta_\nu')\partial_s \nu + \nabla_x \beta . \vec{T}\,, \tag{5.7}$$

$$\partial_t g = -g\langle k\beta\rangle_\Gamma + (L - g)\omega\,, \tag{5.8}$$

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}\,. \tag{5.9}$$

It is worth to note that the strong reaction term $k^2\beta$ in (2.10) has been replaced by the averaged term $k\langle k\beta\rangle_\Gamma$ in (5.6). A similar phenomenon can be observed in (5.8). This is very important feature as it allows for construction of an efficient and stable numerical scheme.

5.1.2. Locally dependent tangential velocity functional. Another possibility for grid points redistribution along evolved curves is based on a tangential velocity functional defined locally. If we take $\alpha = \partial_s \theta$, i.e. $\partial_s \alpha = \partial_s^2 \theta$ then the constitutive equation (5.1) reads as follows: $\partial_t \theta + k\beta - \langle k\beta\rangle_\Gamma = \partial_s^2 \theta$. Since this equation has a parabolic nature one can expect that variations in θ are decreaing during evolution and θ tends to a constant value along the curve Γ due to the diffusion process. The advantage of the particular choice

$$\alpha = \partial_s \theta = \partial_s \ln(g/L) = \partial_s \ln g \tag{5.10}$$

has been already observed by Deckelnick in [**Dec97**]. He analyzed the mean curvature flow of planar curves (i.e. $\beta = k$) by means of a solution to the intrinsic heat equation

$$\partial_t x = \frac{\partial_u^2 x}{|\partial_u x|^2}\,, \quad u \in S^1, t \in (0, T),$$

describing evolution of the position vector x of a curve $\Gamma_t = \mathrm{Img}(x(., t))$. By using Frenét's formulae we obtain $\partial_t x = k\vec{N} + \alpha\vec{T}$ where $\alpha = \partial_s \ln g = \partial_s \ln(g/L) = \partial_s \theta$.

Inserting the tangential velocity functional $\alpha = \partial_s \theta = \partial_s(\ln g)$ into (2.10)–(2.13) we obtain the following system of governing equations:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta, \tag{5.11}$$

$$\partial_t \nu = \beta_k' \partial_s^2 \nu + (\alpha + \beta_\nu')\partial_s \nu + \nabla_x \beta . \vec{T}, \tag{5.12}$$

$$\partial_t g = -gk\beta + g\partial_s^2(\ln g), \tag{5.13}$$

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}\,. \tag{5.14}$$

Notice that equation (5.13) is a nonlinear parabolic equation whereas (5.8) is a nonlocal ODE for the local length g.

5.2. Flowing finite volume approximation scheme

The aim of this part is to review numerical methods for solving the system of equations (2.10)–(2.13). We begin with a simpler case in which we assume the normal velocity to be an affine function of the curvature with coefficients depending on the tangent angle only. Next we consider a slightly generalized form of the normal velocity in which coefficients may also depend on the position vector x.

5.2.0.1. *Normal velocity depending on the tangent angle.* First, we consider a simpler case in which the normal velocity β has the following form:

$$\beta = \beta(k,\nu) = \gamma(\nu)k + F \tag{5.15}$$

with a given anisotropy function $\gamma(\nu) > 0$ and a constant driving force F. The system of governing equations is accompanied by the tangential velocity α given by

$$\partial_s \alpha = k\beta - \frac{1}{L}\int_\Gamma k\beta \mathrm{d}s - \omega(1 - \frac{L}{g}) \tag{5.16}$$

where L is the total length of the curve Γ and ω is a relaxation function discussed in Section 5.1.1. Since there is no explicit dependence of flow on spatial position x the governing equations are simpler and the evolving curve Γ_t is given (uniquely up to a translation) by reconstruction

$$x(u,.) = \int_0^u g\vec{T}\mathrm{d}u = \int_0^s \vec{T}\mathrm{s}. \tag{5.17}$$

Before performing temporal and spatial discretization we insert (5.16) into (2.10) and (2.12) to obtain

$$\partial_t k = \partial_s^2\beta + \partial_s(\alpha k) + k\langle k\beta\rangle + k\omega(1 - \frac{L}{g}), \tag{5.18}$$

$$\partial_t \nu = \beta'_k \partial_s^2 \nu + (\alpha + \beta'_\nu)\partial_s \nu\,, \tag{5.19}$$

$$\partial_t g = -g\langle k\beta\rangle - \omega(g - L). \tag{5.20}$$

From the numerical discretization point of view, critical terms in Eqs. (2.10) – (2.12) are represented by the reaction term $k^2\beta$ in (2.10) and the decay term $k\beta$ in (2.12). In Eqs. (5.18) – (5.20) these critical terms were replaced by the averaged value of $k\beta$ along the curve, thus computation of a local element length in the neighborhood of point with a high curvature is more stable.

In our computational method a solution of the evolution Eq. (1.1) is represented by discrete plane points $x_i^j, i = 0, ..., n,\ j = 0, ..., m$, where index i represents space discretization and index j a discrete time stepping. Since we only consider closed initial curves the periodicity condition $x_0^0 = x_n^0$ is required at the beginning. If we take a uniform division of the time interval $[0,T]$ with a time step $\tau = T/m$ and a uniform division of the fixed parameterization interval $[0,1]$ with a step $h = 1/n$, a point x_i^j corresponds to $x(ih, j\tau)$. Difference equations will be given for discrete quantities k_i^j, ν_i^j, r_i^j, $i = 1, ..., n,\ j = 1, ..., m$ representing piecewise constant approximations of the curvature, tangent angle and element length for the segment $\left[x_{i-1}^j, x_i^j\right]$ and for α_i^j representing tangential velocity of the flowing node x_i^{j-1}. Then, at the j-th discrete time level, $j = 1, ..., m$, approximation of a curve is given by a discrete version of the reconstruction formula (5.17)

$$x_i^j = x_0^j + \sum_{l=1}^{i} r_l^j(\cos(\nu_l^j), \sin(\nu_l^j)), \quad i = 1, ..., n. \tag{5.21}$$

In order to construct a discretization scheme for solving (5.18) – (5.20) we consider time dependent functions $k_i(t)$, $\nu_i(t)$, $r_i(t)$, $x_i(t)$, $\alpha_i(t)$; k_i^j, ν_i^j, r_i^j, x_i^j, α_i^j, described above, represent their values at time levels $t = j\tau$. Let us denote $B = \frac{1}{L}\int_\Gamma k\beta ds$.

We integrate Eqs. (5.16) and (5.18) – (5.20) at any time t over the so-called *flowing control volume* $[x_{i-1}, x_i]$. Using the Newton-Leibniz formula and constant approximation of the quantities inside flowing control volumes, at any time t we get

$$\alpha_i - \alpha_{i-1} = r_i k_i \beta(k_i, \nu_i) - r_i B - \omega \left(r_i - \frac{L}{n} \right).$$

By taking discrete time stepping, for *values of the tangential velocity* α_i^j we obtain

$$\alpha_i^j = \alpha_{i-1}^j + r_i^{j-1} k_i^{j-1} \beta(k_i^{j-1}, \nu_i^{j-1}) - r_i^{j-1} B^{j-1} - \omega(r_i^{j-1} - M^{j-1}), \tag{5.22}$$

$i = 1, ..., n$, with $\alpha_0^j = 0$ (x_0^j is moving only in the normal direction) where

$$M^{j-1} = \frac{1}{n} L^{j-1}, \; L^{j-1} = \sum_{l=1}^{n} r_l^{j-1}, \; B^{j-1} = \frac{1}{L^{j-1}} \sum_{l=1}^{n} r_l^{j-1} k_l^{j-1} \beta(k_l^{j-1}, \nu_l^{j-1})$$

and $\omega = \kappa_1 + \kappa_2 B^{j-1}$, with input redistribution parameters κ_1, κ_2. Using similar approach as above, Eq. (5.20) gives us

$$\frac{dr_i}{dt} + r_i B + r_i \omega = \omega \frac{L}{n}.$$

By taking a backward time difference we obtain an update for local lengths

$$r_i^j = \frac{r_i^{j-1} + \tau \omega M^{j-1}}{1 + \tau(B^{j-1} + \omega)}, \quad i = 1, ..., n, \quad r_0^j = r_n^j, \quad r_{n+1}^j = r_1^j. \tag{5.23}$$

Subsequently, new local lengths are used for approximation of intrinsic derivatives in (5.18) – (5.19). Integrating the curvature Eq. (5.18) in flowing control volume $[x_{i-1}, x_i]$ we have

$$r_i \frac{dk_i}{dt} = [\partial_s \beta(k, \nu)]_{x_{i-1}}^{x_i} + [\alpha k]_{x_{i-1}}^{x_i} + k_i (r_i (B + \omega) - \omega \frac{L}{n}).$$

Now, by replacing the time derivative by time difference, approximating k in nodal points by the average value of neighboring segments, and using semi-implicit approach we obtain a *tridiagonal system* with periodic boundary conditions imposed for new discrete values of the curvature

$$a_i^j k_{i-1}^j + b_i^j k_i^j + c_i^j k_{i+1}^j = d_i^j, \quad i = 1, ..., n, \quad k_0^j = k_n^j, \quad k_{n+1}^j = k_1^j, \tag{5.24}$$

where

$$a_i^j = \frac{\alpha_{i-1}^j}{2} - \frac{\gamma(\nu_{i-1}^{j-1})}{q_{i-1}^j}, \quad c_i^j = -\frac{\alpha_i^j}{2} - \frac{\gamma(\nu_{i+1}^{j-1})}{q_i^j}, \quad d_i^j = \frac{r_i^j}{\tau} k_i^{j-1},$$

$$b_i^j = r_i^j \left(\frac{1}{\tau} - (B^{j-1} + \omega) \right) + \omega M^{j-1} - \frac{\alpha_i^j}{2} + \frac{\alpha_{i-1}^j}{2} + \frac{\gamma(\nu_i^{j-1})}{q_{i-1}^j} + \frac{\gamma(\nu_i^{j-1})}{q_i^j}$$

where $q_i^j = \frac{r_i^j + r_{i+1}^j}{2}$, $i = 1, ..., n$. Finally, by integrating the tangent angle Eq. (5.19) we get

$$r_i \frac{d\nu_i}{dt} = \gamma(\nu_i) \left[\partial_s \nu\right]_{x_{i-1}}^{x_i} + [\alpha \nu]_{x_{i-1}}^{x_i} - \nu_i (\alpha_i - \alpha_{i-1}) + \gamma'(\nu_i) k_i \left[\nu\right]_{x_{i-1}}^{x_i}.$$

Again, values of the tangent angle ν in nodal points are approximated by the average of neighboring segments values, the time derivative is replaced by the time difference and using a semi-implicit approach we obtain *tridiagonal system* with periodic boundary conditions *for new values of the tangent angle*

$$A_i^j \nu_{i-1}^j + B_i^j \nu_i^j + C_i^j \nu_{i+1}^j = D_i^j, \quad i = 1, ..., n, \quad \nu_0^j = \nu_n^j, \quad \nu_{n+1}^j = \nu_1^j, \tag{5.25}$$

where

$$A_i^j = \frac{\alpha_{i-1}^j}{2} + \frac{\gamma'(\nu_i^{j-1}) k_i^j}{2} - \frac{\gamma(\nu_i^{j-1})}{q_{i-1}^j}, \quad B_i^j = \frac{r_i^j}{\tau} - (A_i^j + C_i^j),$$

$$C_i^j = -\frac{\alpha_i^j}{2} - \frac{\gamma'(\nu_i^{j-1}) k_i^j}{2} - \frac{\gamma(\nu_i^{j-1})}{q_i^j}, \quad D_i^j = \frac{r_i^j}{\tau} \nu_i^{j-1}.$$

The initial quantities for the algorithm are computed as follows:

$$R_i = (R_{i_1}, R_{i_2}) = x_i^0 - x_{i-1}^0, \quad i = 1, ..., n, \quad R_0 = R_n, \quad R_{n+1} = R_1,$$

$$r_i^0 = |R_i|, \quad i = 0, ..., n+1, \tag{5.26}$$

$$k_i^0 = \tfrac{1}{2r_i^0} \mathrm{sgn}\left(\det(R_{i-1}, R_{i+1})\right) \arccos\left(\tfrac{R_{i+1}.R_{i-1}}{r_{i+1}^0 r_{i-1}^0}\right), \tag{5.27}$$

$$i = 1, ..., n, \; k_0^0 = k_n^0, \; k_{n+1}^0 = k_1^0,$$

$$\nu_0^0 = \arccos(R_{i_1}/r_i^0) \text{ if } R_{i_2} \geq 0, \quad \nu_0^0 = 2\pi - \arccos(R_{i_1}/r_i) \text{ if } R_{i_2} < 0,$$

$$\nu_i^0 = \nu_{i-1}^0 + r_i^0 k_i^0, \quad i = 1, ..., n+1. \tag{5.28}$$

Remark (*Solvability and stability of the scheme.*) Let us first examine discrete values of the tangent angle ν computed from (5.25). One can rewrite it into the form

$$\nu_i^j + \frac{\tau}{r_i^j} C_i^j (\nu_{i+1}^j - \nu_i^j) + \frac{\tau}{r_i^j} A_i^j (\nu_{i-1}^j - \nu_i^j) = \nu_i^{j-1}. \tag{5.29}$$

Let $\max_k \nu_k^j$ be attained at the i-th node. We can always take a fine enough resolution of the curve, i.e. take small $q_i^j \ll 1$, $i = 1, .., n$, such that both A_i^j and C_i^j are nonpositive and thus the second and third terms on the left hand side of (5.29) are nonnegative. Then $\max_k \nu_k^j = \nu_i^j \leq \nu_i^{j-1} \leq \max_k \nu_k^{j-1}$. By a similar argument we can derive an inequality for the minimum. In this way we have shown the L^∞-stability criterion, namely

$$\min_k \nu_k^0 \leq \min_k \nu_k^j \leq \max_k \nu_k^j \leq \max_k \nu_k^0, \quad j = 1, .., m.$$

Notice that in the continuous case the above comparison inequality is a consequence of the parabolic maximum principle for equation (5.7) in which the term $\nabla_x \beta . \vec{T}$ is vanishing as β does not explicitly depend on the position vector x.

Having guaranteed non-positivity of A_i^j and C_i^j we can conclude positivity and diagonal dominance of the diagonal term B_i^j. In particular, it implies that the tridiagonal matrix of the system (5.25) is an M-matrix and hence a solution to (5.25) always exists and is unique.

In the same way, by taking q_i^j small enough, we can prove nonpositivity of the off-diagonal terms a_i^j and c_i^j in the system (5.24) for discrete curvature values. Then the diagonal term b_i^j is positive and dominant provided that $\tau(B^{j-1} + \omega) < 1$.

Again we have shown that the corresponding matrix is an M-matrix and therefore there exists a unique solution to the system (5.24).

Another natural stability requirement of the scheme is related to the positivity of local lengths r_i^j during computations. It follows from (5.23) that the positivity of r_i^j is equivalent to the condition $\tau(B^{j-1}+\omega) > -1$. Taking into account both inequalities for the time step we end up with the following stability restriction on the time step τ:

$$\tau \leq \frac{1}{|B^{j-1}+\omega|} \tag{5.30}$$

related to B^{j-1} (a discrete average value of $k\beta$ over a curve).

5.2.0.2. *Normal velocity depending on the tangent angle and the position vector.* Next we consider a more general motion of the curves with explicit dependence of the flow on position x and suggest numerical scheme for such a situation. We consider (1.1) with a linear dependence of β on the curvature, i.e.

$$\beta(k,x,\nu) = \delta(x,\nu)k + c(x,\nu)$$

where $\delta(x,\nu) > 0$. By using Frenét's formulae one can rewrite the position vector Eq. (2.13) as an intrinsic convection-diffusion equation for the vector x and we get the system

$$\partial_t k = \partial_s^2\beta + \partial_s(\alpha k) + k\frac{1}{L}\int_\Gamma k\beta ds + k\omega(1-\frac{L}{g})\,, \tag{5.31}$$

$$\partial_t \nu = \beta'_k\partial_s^2\nu + (\alpha+\beta'_\nu)\partial_s\nu + \nabla_x\beta.\vec{T}\,, \tag{5.32}$$

$$\partial_t g = -g\frac{1}{L}\int_\Gamma k\beta ds - \omega(g-L)\,, \tag{5.33}$$

$$\partial_t x = \delta(x,\nu)\partial_s^2 x + \alpha\partial_s x + \vec{c}(x,\nu)\,, \tag{5.34}$$

where $\vec{c}(x,\nu) = c(x,\nu)\vec{N} = (-c(x,\nu)\sin\nu, c(x,\nu)\cos\nu)$. In comparison to the scheme given above, two new tridiagonal systems have to be solved at each time level in order to update the curve position vector x. The curve position itself and all geometric quantities entering the model are resolved from their own intrinsic Eqs. (5.31) – (5.34). In order to construct a discretization scheme, Eqs. (5.31) – (5.33) together with (5.16) are integrated over a flowing control volume $[x_{i-1}, x_i]$. We also construct a time dependent dual volumes $\left[\tilde{x}_{i-1}^j, \tilde{x}_i^j\right]$, $i = 1,..,n, j = 1,..,m$, where $\tilde{x}_i^j = \frac{x_{i-1}^j + x_i^j}{2}$ over which the last Eq. (5.34) will be integrated. Then, for *values of the tangential velocity* we obtain

$$\alpha_i^j = \alpha_{i-1}^j + r_i^{j-1}k_i^{j-1}\beta(\tilde{x}_i^{j-1}, k_i^{j-1}, \nu_i^{j-1}) - r_i^{j-1}B^{j-1} - \omega(r_i^{j-1} - M^{j-1}),$$
$$i = 1,...,n, \quad \alpha_0^j = 0, \tag{5.35}$$

with M^{j-1}, L^{j-1}, ω given as above and

$$B^{j-1} = \frac{1}{L^{j-1}}\sum_{l=1}^{n} r_l^{j-1}k_l^{j-1}\beta(\tilde{x}_l^{j-1}, k_l^{j-1}, \nu_l^{j-1}).$$

Local lengths are updated by the formula:

$$r_i^j = \frac{r_i^{j-1} + \tau\omega M^{j-1}}{1+\tau(B^{j-1}+\omega)}, \quad i=1,...,n, \quad r_0^j = r_n^j, \quad r_{n+1}^j = r_1^j. \tag{5.36}$$

The *tridiagonal system for discrete values of the curvature* reads as follows:

$$a_i^j k_{i-1}^j + b_i^j k_i^j + c_i^j k_{i+1}^j = d_i^j, \quad i=1,...,n, \quad k_0^j = k_n^j, \quad k_{n+1}^j = k_1^j, \tag{5.37}$$

where

$$a_i^j = \frac{\alpha_{i-1}^j}{2} - \frac{\delta(\tilde{x}_{i-1}^{j-1}, \nu_{i-1}^{j-1})}{q_{i-1}^j}, \quad c_i^j = -\frac{\alpha_i^j}{2} - \frac{\delta(\tilde{x}_{i+1}^{j-1}, \nu_{i+1}^{j-1})}{q_i^j},$$

$$b_i^j = r_i^j\left(\frac{1}{\tau} - (B^{j-1}+\omega)\right) + \omega M^{j-1} - \frac{\alpha_i^j}{2} + \frac{\alpha_{i-1}^j}{2} + \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_{i-1}^j} + \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_i^j},$$

$$d_i^j = \frac{r_i^j}{\tau}k_i^{j-1} + \frac{c(\tilde{x}_{i+1}^{j-1}, \nu_{i+1}^{j-1}) - c(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_i^j} - \frac{c(\tilde{x}_i^{j-1}, \nu_i^{j-1}) - c(\tilde{x}_{i-1}^{j-1}, \nu_{i-1}^{j-1})}{q_{i-1}^j}.$$

The *tridiagonal system for new values of the tangent angle* is given by

$$A_i^j \nu_{i-1}^j + B_i^j \nu_i^j + C_i^j \nu_{i+1}^j = D_i^j, \quad i=1,...,n, \quad \nu_0^j = \nu_n^j, \quad \nu_{n+1}^j = \nu_1^j, \tag{5.38}$$

where

$$A_i^j = \frac{\alpha_{i-1}^j + \beta'_\nu(\tilde{x}_i^{j-1}, k_i^j, \nu_i^{j-1})}{2} - \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_{i-1}^j},$$

$$C_i^j = -\frac{\alpha_i^j + \beta'_\nu(\tilde{x}_i^{j-1}, k_i^j, \nu_i^{j-1})}{2} - \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_i^j},$$

$$B_i^j = \frac{r_i^j}{\tau} - (A_i^j + C_i^j), \quad D_i^j = \frac{r_i^j}{\tau}\nu_i^{j-1} + r_i^j \nabla_x \beta(\tilde{x}_i^{j-1}, \nu_i^{j-1}, k_i^j).(\cos(\nu_i^{j-1}), \sin(\nu_i^{j-1})).$$

Finally, we end up with *two tridiagonal systems for updating the position vector*

$$\mathcal{A}_i^j x_{i-1}^j + \mathcal{B}_i^j x_i^j + \mathcal{C}_i^j x_{i+1}^j = \mathcal{D}_i^j, \quad i=1,...,n, \quad x_0^j = x_n^j, \quad x_{n+1}^j = x_1^j, \tag{5.39}$$

where

$$\mathcal{A}_i^j = -\frac{\delta(\tilde{x}_i^{j-1}, \frac{1}{2}(\nu_i^j + \nu_{i+1}^j))}{r_i^j} + \frac{\alpha_i^j}{2}, \quad \mathcal{C}_i^j = -\frac{\delta(\tilde{x}_i^{j-1}, \frac{1}{2}(\nu_i^j + \nu_{i+1}^j))}{r_{i+1}^j} - \frac{\alpha_i^j}{2},$$

$$\mathcal{B}_i^j = \frac{q_i^j}{\tau} - (\mathcal{A}_i^j + \mathcal{C}_i^j), \quad \mathcal{D}_i^j = \frac{q_i^j}{\tau}x_i^{j-1} + q_i^j \vec{c}(x_i^{j-1}, \frac{1}{2}(\nu_i^j + \nu_{i+1}^j)).$$

The *initial quantities* for the algorithm are given by (5.26) – (5.28).

CHAPTER 6

Applications of curvature driven flows

6.1. Computation of curvature driven evolution of planar curves with external force

In following figures we present numerical solutions computed by the scheme; initial curves are plotted with a thick line and the numerical solution is given by further solid lines with points representing the motion of some grid points during the curve evolution. In Figure 1 we compare computations with and without tangential redistribution for a large driving force F. As an initial curve we chose $x_1(u) = \cos(2\pi u)$, $x_2(u) = 2\sin(2\pi u) - 1.99\sin^3(2\pi u)$, $u \in [0,1]$. Without redistribution, the computations are collapsing soon because of the degeneracy in local element lengths in parts of a curve with high curvature leading to a merging of the corresponding grid points. Using the redistribution the evolution can be successfully handled. We used $\tau = 0.00001$, 400 discrete grid points and we plotted every $150th$ time step. In Figure 2 we have considered an initial curve $x_1(u) = (1 - C\cos^2(2\pi u))\cos(2\pi u)$, $x_2(u) = (1 - C\cos^2(2\pi u))\sin(2\pi u)$, $u \in [0,1]$ with $C = 0.7$. We took $\tau = 0.00001$ and 800 (Figure 2 left) and 1600 (Figure 2 right) grid points for representation of a curve. In Figure 2 left we plot each $500th$ time step, and in Figure 2 right each $100th$ step. It is natural that we have to use small time steps in case of strong driving force. However, the time step is not restricted by the point-wise values of the almost singular curvature in the corners which would lead to an un-realistic time step restriction. According to (5.30), the time step is restricted by the average value of $k\beta$ computed over the curve which is much more weaker restriction because of the regularity of the curve outside the corners. In Figure 3 we present experiments with three-fold anisotropy starting with unit circle. We used $\tau = 0.001$, 300 grid points and we plotted every $50th$ time step (left) and every $750th$ time step (right). In all experiments we chose redistribution parameters $\kappa_1 = \kappa_2 = 10$.

6.2. Flows of curves on a surface driven by the geodesic curvature

The purpose of this section is to analytically and numerically investigate a flow of closed curves on a given graph surface driven by the geodesic curvature. We show how such a flow can be reduced to a flow of vertically projected planar curves governed by a solution of a fully nonlinear system of parabolic differential equations. We present various computational examples of evolution of surface curves driven by the geodesic curvature are presented in this part. The normal velocity $\mathcal{V}$ of the evolving family of surface curves $\mathcal{G}^t, t \geq 0$, is proportional to the geodesic curvature

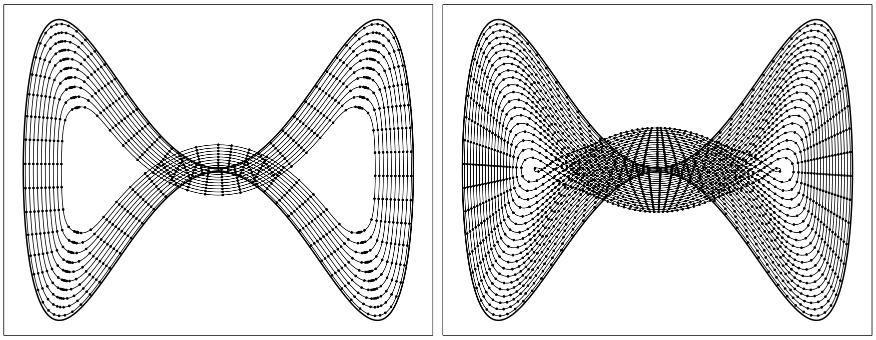

FIGURE 1. Isotropic curvature driven motion, $\beta(k,\nu) = \varepsilon k + F$, with $\varepsilon = 1$, $F = 10$, without (left) and with (right) uniform tangential redistribution of grid points.

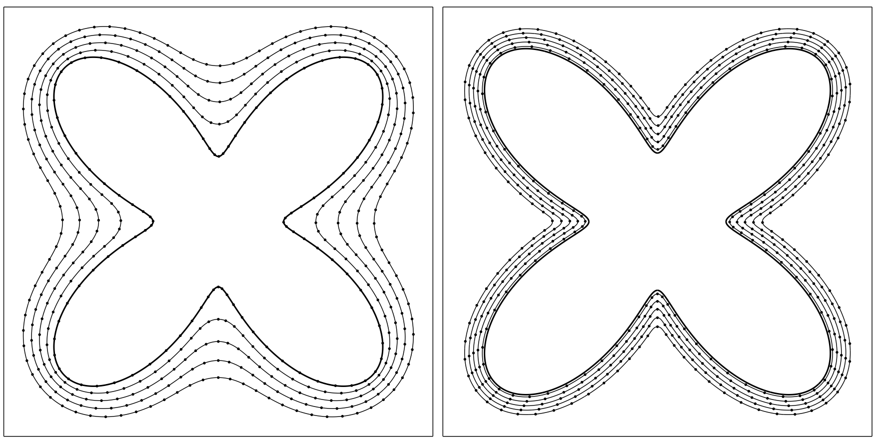

FIGURE 2. Isotropic curvature driven motion of an initial nonconvex curve including uniform tangential redistribution of grid points; $\beta(k,\nu) = \varepsilon k + F$, with $\varepsilon = 1$, $F = -10$ (left) and $\varepsilon = 0.1$, $F = -10$ (right). Resolution of sharp corners in the case of a highly dominant forcing term using the algorithm with redistribution is possible.

$\mathcal{K}_g$ of $\mathcal{G}^t$, i.e.

$$\mathcal{V} = \delta \mathcal{K}_g \tag{6.1}$$

where $\delta = \delta(X, \vec{\mathcal{N}}) > 0$ is a smooth positive coefficient describing anisotropy depending on the position X and the orientation of the unit inward normal vector $\vec{\mathcal{N}}$ to the curve on a surface.

The idea how to analyze and compute numerically such a flow is based on the so-called direct approach method applied to a flow of vertically projected family of planar curves. Vertical projection of surface curves on a simple surface $\mathcal{M}$ into the

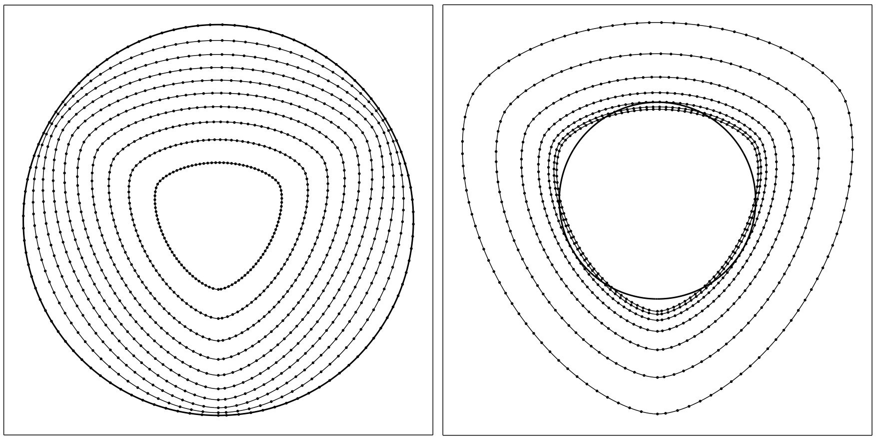

FIGURE 3. Anisotropic curvature driven motion of the initial unit circle including uniform tangential redistribution of grid points; $\beta(k,\nu) = \gamma(\nu)k + F$, with $\gamma(\nu) = 1 - \frac{7}{9}\cos(3\nu)$, $F = 0$ (left) and $\gamma(\nu) = 1 - \frac{7}{9}\cos(3\nu)$, $F = -1$ (right).

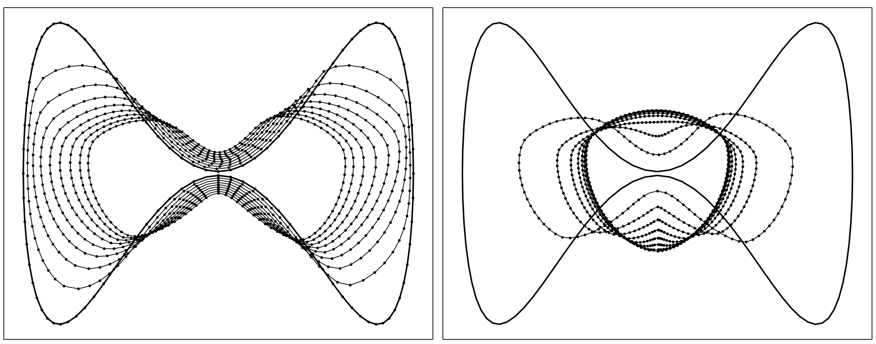

FIGURE 4. Curve evolution governed by $v = (1 - \frac{8}{9}\cos(3\nu))(x_1^2 + x_2^2)\ k + (-x_1, -x_2).(-\sin\nu, \cos\nu) - 0.5$.

plane $\mathbb{R}^2$. It allows for reducing the problem to the analysis of evolution of planar curves $\Gamma^t : S^1 \to \mathbb{R}^2$, $t \geq 0$ driven by the normal velocity v given as a nonlinear function of the position vector x, tangent angle ν and as an affine function of the curvature k of Γ^t, i.e.

$$v = \beta(x, \nu, k) \tag{6.2}$$

where $\beta(x,\nu,k) = a(x,\nu)k + c(x,\nu)$ and $a(x,\nu) > 0, c(x,\nu)$ are bounded smooth coefficients.

6.2.1. Planar projection of the flow on a graph surface. Throughout this section we will always assume that a surface $\mathcal{M} = \{(x,z)) \in \mathbb{R}^3,\ z = \phi(x),\ x \in$

$\Omega\}$ is a smooth graph of a function $\phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ defined in some domain $\Omega \subset \mathbb{R}^2$. Hereafter, the symbol (x, z) stands for a vector $(x_1, x_2, z) \in \mathbb{R}^3$ where $x = (x_1, x_2) \in \mathbb{R}^2$. In such a case any smooth closed curve $\mathcal{G}$ on the surface $\mathcal{M}$ can be then represented by its vertical projection to the plane, i.e. $\mathcal{G} = \{(x, z) \in \mathbb{R}^3,\ x \in \Gamma,\ z = \phi(x)\}$ where Γ is a closed planar curve in $\mathbb{R}^2$. Recall, that for a curve $\mathcal{G} = \{(x, \phi(x)) \in \mathbb{R}^3,\ x \in \Gamma\}$ on a surface $\mathcal{M} = \{(x_1, x_2, \phi(x_1, x_2)) \in \mathbb{R}^3,\ (x_1, x_2) \in \Omega\}$ the geodesic curvature $\mathcal{K}_g$ is given by

$$\mathcal{K}_g = -\sqrt{EG - F^2}\Big(x_1'' x_2' - x_1' x_2'' - \Gamma^2_{11} x_1'^3 + \Gamma^1_{22} x_2'^3 \\ -(2\Gamma^2_{12} - \Gamma^1_{11}) x_1'^2 x_2' + (2\Gamma^1_{12} - \Gamma^2_{22}) x_1' x_2'^2 \Big)$$

where E, G, F are coefficients of the first fundamental form and Γ^k_{ij} are Christoffel symbols of the second kind. Here $(.)'$ denotes the derivative with respect to the unit speed parameterization of a curve on a surface. In terms of geometric quantities related to a vertically projected planar curve we obtain, after some calculations, that

$$\mathcal{K}_g = \frac{1}{\left(1 + (\nabla\phi.\vec{T})^2\right)^{\frac{3}{2}}} \left(\left(1 + |\nabla\phi|^2\right)^{\frac{1}{2}}\ k + \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{(1 + |\nabla\phi|^2)^{\frac{1}{2}}} \nabla\phi.\vec{N} \right) \tag{6.3}$$

(see [**MS04b**]). Moreover, the unit inward normal vector $\vec{\mathcal{N}} \perp T_x(\mathcal{M})$ to a surface curve $\mathcal{G} \subset \mathcal{M}$ relative to $\mathcal{M}$ can be expressed as

$$\vec{\mathcal{N}} = \frac{\left((1 + (\nabla\phi.\vec{T})^2)\vec{N} - (\nabla\phi.\vec{T})(\nabla\phi.\vec{N})\vec{T},\ \nabla\phi.\vec{N} \right)}{\left((1 + |\nabla\phi|^2)(1 + (\nabla\phi.\vec{T})^2) \right)^{\frac{1}{2}}}$$

(see also [**MS04b**]). Hence for the normal velocity $\mathcal{V}$ of $\mathcal{G}_t = \{(x, \phi(x)),\ x \in \Gamma^t\}$ we have

$$\mathcal{V} = \partial_t(x, \phi(x)).\vec{\mathcal{N}} = (\vec{N}, \nabla\phi.\vec{N}).\beta\vec{\mathcal{N}} = \left(\frac{1 + |\nabla\phi|^2}{1 + (\nabla\phi.\vec{T})^2} \right)^{\frac{1}{2}} \beta$$

where β is the normal velocity of the vertically projected planar curve Γ^t having the unit inward normal $\vec{N}$ and tangent vector $\vec{T}$. Following the so-called direct approach (see [**Dec97, Dzi94, Dzi99, HLS94, Mik97, MS99, MS01, MS04a, MS04b, MS06**]) the evolution of planar curves $\Gamma^t, t \geq 0$, can be described by a solution $x = x(., t) \in \mathbb{R}^2$ to the position vector equation $\partial_t x = \beta\vec{N} + \alpha\vec{T}$ where β and α are normal and tangential velocities of Γ^t, resp. Assuming the family of surface curves $\mathcal{G}_t$ satisfies (6.1) it has been shown in [**MS04b**] that the geometric equation $v = \beta(x, k, \nu)$ for the normal velocity v of the vertically projected planar curve Γ^t can be written in the following form:

$$v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla\phi(x).\vec{N} \tag{6.4}$$

where $a = a(x, \nu) > 0$ and $b = b(x, \nu)$ are smooth functions given by

$$a(x, \nu) = \frac{\delta}{1 + (\nabla\phi.\vec{T})^2}\,, \quad b(x, \nu) = -a(x, \nu) \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{1 + |\nabla\phi|^2}\,, \tag{6.5}$$

where $\delta(X, \vec{\mathcal{N}}) > 0, X = (x, \phi(x)), \phi = \phi(x)$, k is the curvature of Γ^t, and $\vec{N} = (-\sin\nu, \cos\nu)$ and $\vec{T} = (\cos\nu, \sin\nu)$ are the unit inward normal and tangent vectors to a curve Γ^t.

We can also consider a more general flow of curves on a given surface driven by the normal velocity

$$\mathcal{V} = \mathcal{K}_g + \mathcal{F} \tag{6.6}$$

where $\mathcal{F}$ is the normal component of a given external force $\vec{G}$, i.e. $\mathcal{F} = \vec{G}.\vec{\mathcal{N}}$. The external vector field $\vec{G}$ is assumed to be perpendicular to the plane $\mathbb{R}^2$ and it may depend on the vertical coordinate $z = \phi(x)$ only, i.e.

$$\vec{G}(x) = -(0, 0, \gamma)$$

where $\gamma = \gamma(z) = \gamma(\phi(x))$ is a given scalar "gravity" functional.

Assuming the family of surface curves $\mathcal{G}^t$ satisfies (6.6) it has been shown in [**MS04b**] that the geometric equation $v = \beta(x, k, \nu)$ for the normal velocity v of the vertically projected planar curve Γ^t can be written in the following form:

$$v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla\phi(x).\vec{N}$$

where $a = a(x, \nu) > 0$ and $b = b(x, \nu)$ are smooth functions given by

$$a(x, \nu) = \frac{1}{1 + (\nabla\phi.\vec{T})^2}, \quad b(x, \nu) = a(x, \nu)\left(\gamma(\phi) - \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{1 + (\nabla\phi.\vec{T})^2}\right), \tag{6.7}$$

In order to compute evolution of surface curves driven by the geodesic curvature and external force we can use numerical approximation scheme developed in Chapter 5 for the flow of vertically projected planar curves driven by the normal velocity given as in (6.4).

The next couple of examples illustrate a geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps. In Fig. 5 we show an example of an evolving family of surface curves shrinking to a point in finite time. In this example the behavior of evolution of surface curve is similar to that of planar curves for which Grayson's theorem holds. On the other hand, in Fig. 6 we present the case when the surface has two sufficiently high humps preventing evolved curve to pass through them. As it can be seen from Fig. 6 the evolving family of surface curves approaches a closed geodesic curve $\bar{\mathcal{G}}$ as $t \to \infty$.

The initial curve with large variations in the curvature is evolved according to the normal velocity $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ where the external force $\mathcal{F} = \vec{G}.\vec{\mathcal{N}}$ is the normal projection of $\vec{G} = -(0, 0, \gamma)$ (see Fig. 7). In the numerical experiment we considered a strong external force coefficient $\gamma = 30$. The evolving family of surface curves approaches a stationary curve $\bar{\Gamma}$ lying in the bottom of the sharp narrow valley.

In the examples shown in Fig. 8 we present numerical results of simulations of a surface flow driven by the geodesic curvature and gravitational like external force, $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$, on a wave-let surface given by the graph of the function $\phi(x) = f(|x|)$ where $f(r) = \sin(r)/r$ and $\gamma = 2$. In the first example shown in Fig. 8 (left-up) we started from the initial surface curve having large variations in the geodesic curvature. The evolving family converges to the stable stationary curve $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ with the second smallest stable radius. Vertical projection of the evolving family to the plane driven by the normal velocity $v = \beta(x, k, \nu)$ is shown

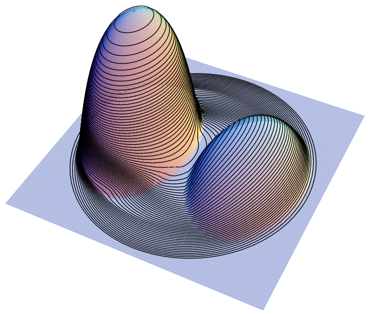

FIGURE 5. A geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps having different heights.

in Fig. 8 (right-up). In Fig. 8 (left-bottom) we study a surface flow on the same surface the same external force. The initial curve is however smaller compared to that of the previous example. In this case the evolving family converges to the stable stationary curve with the smallest stable radius.

6.3. Applications in the theory of image segmentation

6.3.1. Edge detection in static images. A similar equation to (1.1) arises from the theory of image segmentation in which detection of object boundaries in the analyzed image plays an important role. A given black and white image can be represented by its intensity function $I : R^2 \to [0, 255]$. The aim is to detect edges of the image, i.e. closed planar curves on which the gradient ∇I is large (see [**KM95**]). The method of the so-called active contour models is to construct an evolving family of plane curves converging to an edge (see [**KWT87**]).

One can construct a family of curves evolved by the normal velocity $v = \beta(k, x, \nu)$ of the form

$$\beta(k, x, \nu) = \delta(x, \nu)k + c(x, \nu)$$

where $c(x, \nu)$ is a driving force and $\delta(x, \nu) > 0$ is a smoothing coefficient. These functions depend on the position vector x as well as orientation angle ν of a curve. Evolution starts from an initial curve whcih is a suitable approximation of the edge and then it converges to the edge. If $c > 0$ then the driving force shrinks the curve whereas the impact of c is reversed in the case $c < 0$. Let us consider an auxiliary function $\phi(x) = h(|\nabla I(x)|)$ where h is a smooth edge detector function like e.g. $h(s) = 1/(1 + s^2)$. The gradient $-\nabla\phi(x)$ has the important geometric property: it points towards regions where the norm of the gradient ∇I is large (see Fig. 9 right). Let us therefore take $c(x, \nu) = -b(\phi(x))\nabla\phi(x).\vec{N}$ and $\delta(x, \nu) = a(\phi(x))$ where $a, b > 0$ are given smooth functions. Now, if an initial curve belongs to a neighborhood of an edge of the image and it is evolved according to the geometric equation

$$v = \beta(x, k, \nu) \equiv a(\phi(x))k - b(\phi(x))\nabla\phi.\vec{N}$$

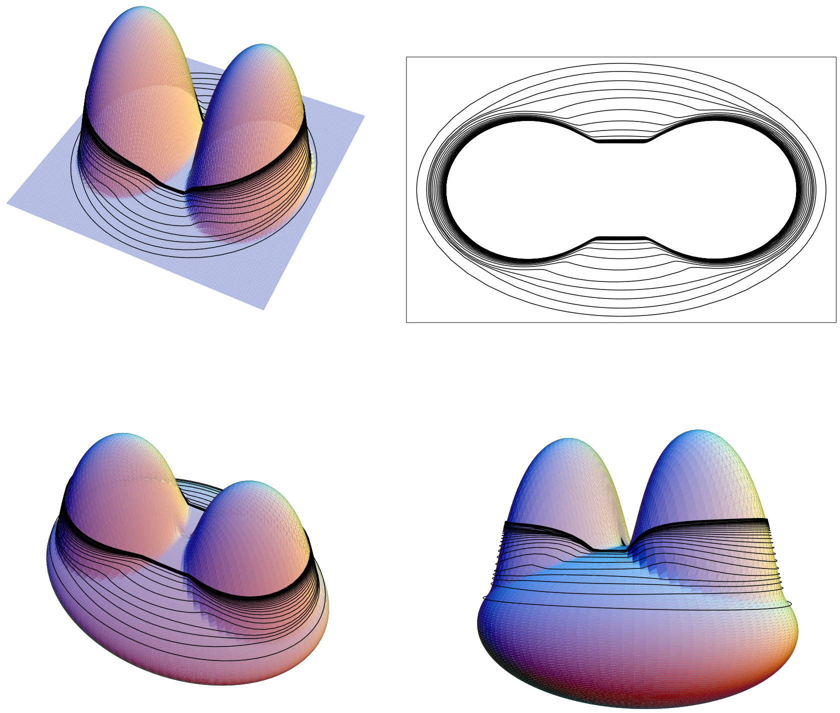

FIGURE 6. A geodesic flow on a surface with two sufficiently high humps (left-up) and its vertical projection to the plane (right-up). The evolving family of surface curves approaches a closed geodesic as $t \to \infty$. The same phenomenon of evolution on a compact manifold without boundary (below).

then it is driven towards this edge. In the context of level set methods, edge detection techniques based on this idea were first discussed by Caselles et al. and Malladi et al. in [**CCCD93, MSV95**] (see also [**CKS97, CKSS97, KKO$^+$96**]).

We apply our computational method to the image segmentation problem. First numerical experiment is shown in Fig. 10. We look for an edge in a 2D slice of a real 3D echocardiography which was prefiltered by the method of [**SMS99**]. The testing data set (the image function I) is a courtesy of Prof. Claudio Lamberti, DEIS, University of Bologna. We have inserted an initial ellipse into the slice close to an expected edge (Fig. 10 left). Then it was evolved according to the normal velocity described above using the time stepping $\tau = 0.0001$ and nonlocal redistribution strategy from Chaper 5. with parameters $\kappa_1 = 20$, $\kappa_2 = 1$ until the

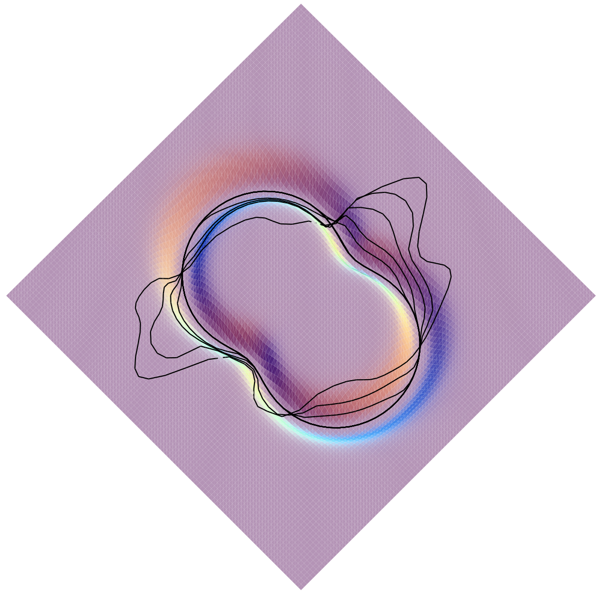

FIGURE 7. A geodesic flow on a flat surface with a sharp narrow valley.

limiting curve has been formed (400 time steps). The final curve representing the edge in the slice can be seen in Fig. 10 right.

Next we present results for the image segmentation problem computed by means of a geodesic flow with external force discussed in Section 6.3. We consider an artificial dumb-bell image.from Fig. 9. If we take $\phi(x) = 1/(1 + |\nabla I(x)|^2)$ then the surface $\mathcal{M}$ defined as a graph of ϕ has a sharp narrow valley corresponding to points of the image in which the gradient $|\nabla I(x)|$ is very large representing thus an edge in the image. In contrast to the previous example shown in Fig. 10 we will make use of the flow of curves on a surface $\mathcal{M}$ driven by the geodesic curvature and strong "gravitational-like" external force $\mathcal{F}$. According to section 6.3 such a surface flow can be represented by a family of vertically projected plane curves driven by the normal velocity

$$v = a(x, \nu)k - b(x, \nu)\nabla\phi(x).\vec{N}$$

where coefficients a, b are defined as in (6.5) with strong external force coefficient $\gamma = 100$. Results of computation are presented in Fig. 11.

6.3.2. Tracking moving boundaries. In this section we describe a model for tracking boundaries in a sequence of moving images. Similarly as in the previous section the model is based on curvature driven flow with an external force depending on the position vector x.

Parametric active contours have been used extensively in computer vision for different tasks like segmentation and tracking. However, all parametric contours are known to suffer from the problem of frequent bunching and spacing out of curve

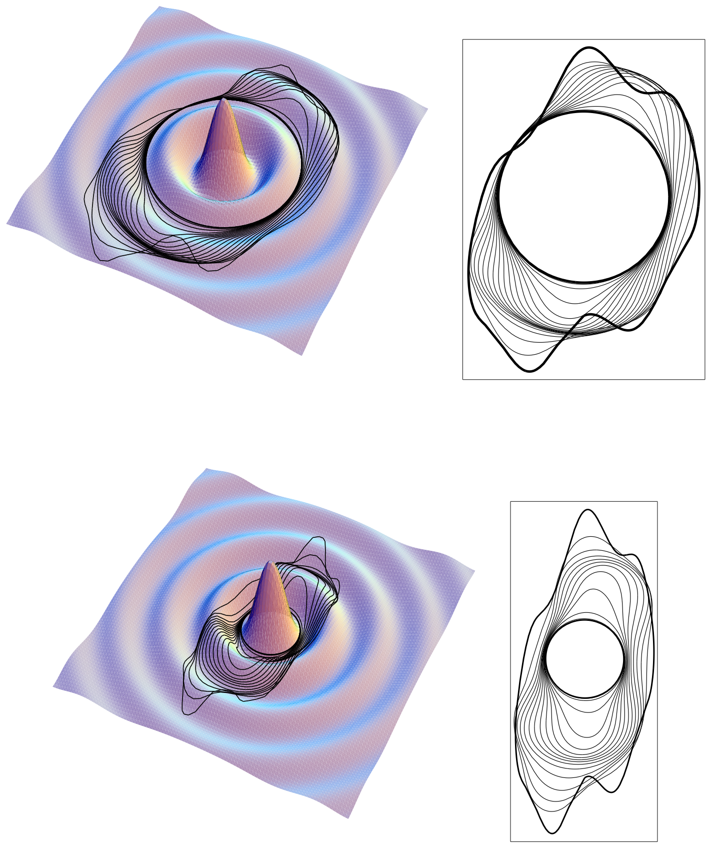

FIGURE 8. A surface flow on a wavelet like surface (left) and its vertical projection to the plane (right). Surface curves converge to the stable stationary circular curve $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ with the smallest stable radius $\bar{r}$ (bottom) and the second smallest radius (up).

points locally during the curve evolution. In this part, we discuss a mathematical basis for selecting such a suitable tangential component for stabilization. We demonstrate the usefulness of the proposed choice of a tangential velocity method with a number of experiments. The results in this section can be found in a recent papers by Srikrishnan et al. [**SCDR07, SCDRS07**].

The force at each point on the curve can be resolved into two components: along the local tangent and normal denoted by α and β, respectively. This is written as:

$$\frac{\partial x}{\partial t} = \beta \vec{N} + \alpha \vec{T} \,. \tag{6.8}$$

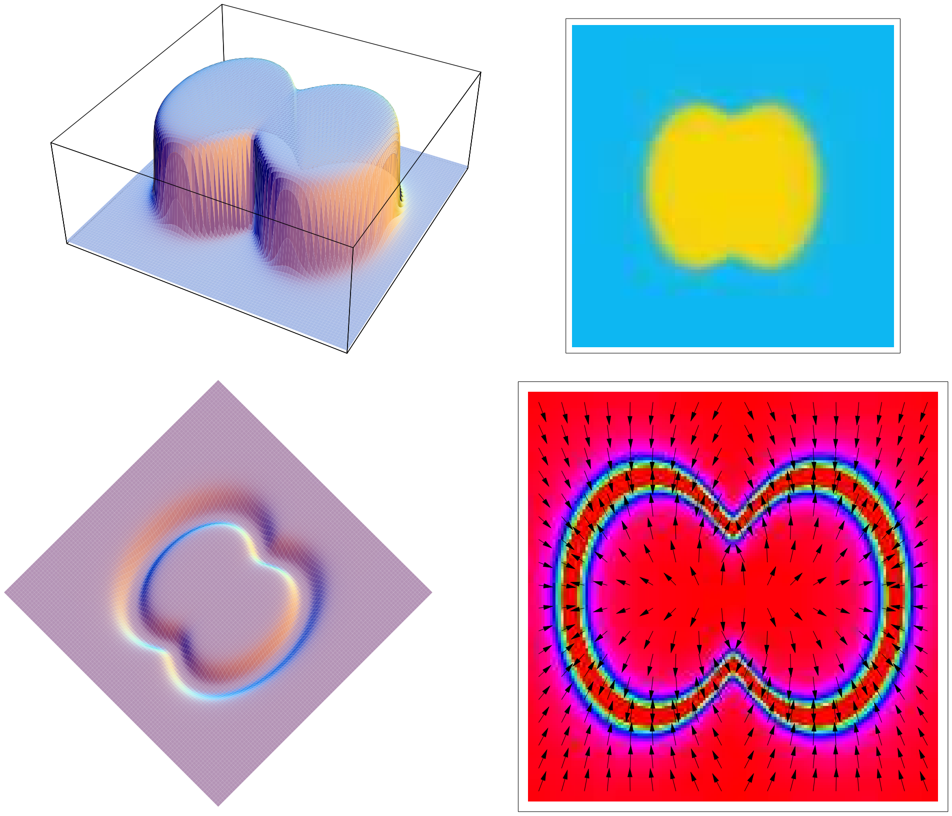

FIGURE 9. An image intensity function $I(x)$ (left-up) corresponding to a "dumb-bell" image (right-up). The the function ϕ (bottom-left) and corresponding vector field $-\nabla\phi(x)$ (bottom-right).

In this application, the normal velocity β has the form: $\beta = \mu\kappa + f(x)$ where f is a bounded function depending on the position of a curve point x. For the purpose of tracking we use the function $f(x) = \log\left(\frac{Prob_B(I(x))}{Prob_T(I(x))}\right)$ and we smoothly cut-off this function if either $Prob_B(I(x))$ or $Prob_B(I(x))$ are less than a prescribed tolerance. Here $Prob_B(I(x))$ stands for the probability that the point x belongs to a background of the image represented by the image intensity function I whereas $Prob_T(I(x))$ represents the probability that the point x belongs to a target in the image to be tracked. Both probabilities can be calculated from the image histogram (see [**SCDR07, SCDRS07**] for details).

In this field of application of a curvature driven flow of planar curves representing tracked boundaries in moving images it is very important to propose a suitable tangential redistribution of numerically computed grid points. Let us demonstrate the importance of tangential velocity by the following motivational example. In Fig. 12, we show two frames from a tracking sequence of a hand. Without any

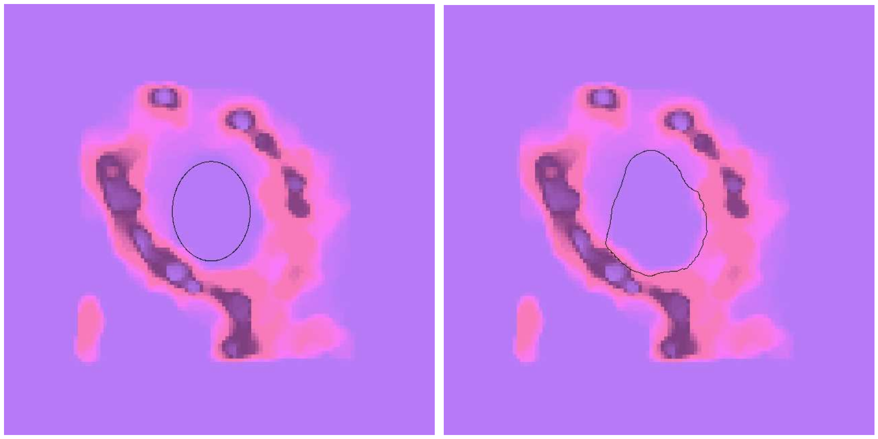

FIGURE 10. An initial ellipse is inserted into the 2D slice of a prefiltered 3D echocardiography (left), the slice together with the limiting curve representing the edge (right).

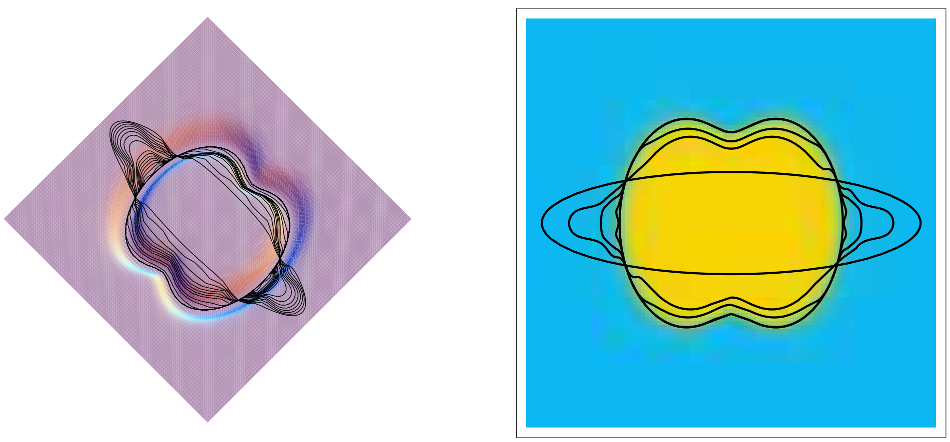

FIGURE 11. A geodesic flow on a flat surface with a sharp narrow valley (left) and its vertical projection to the plane with density plot of the image intensity function $I(x)$ (right).

tangential velocity (i.e. $\alpha = 0$) one can observe formation of small loops in the right picture which is a very next frame to the initial left one. These loops blow up and the curve becomes unstable within the next few frames.

In [**SCDRS07**] we proposed a suitable tangential velocity functional α capable of preventing evolved family of curves (image contours) from formation such undesirable loops like in Fig. 12 (right). Using a tangential velocity satisfying

$$\frac{\partial \alpha}{\partial u} = K - g + g\kappa\beta.$$

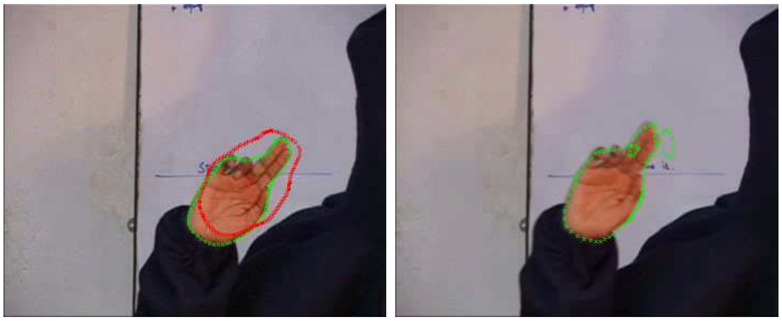

FIGURE 12. Illustration of curve degeneration. Left: The initial curve in red. Right: Bunching of points (in red) starts due to target motion leading to a loop formation.

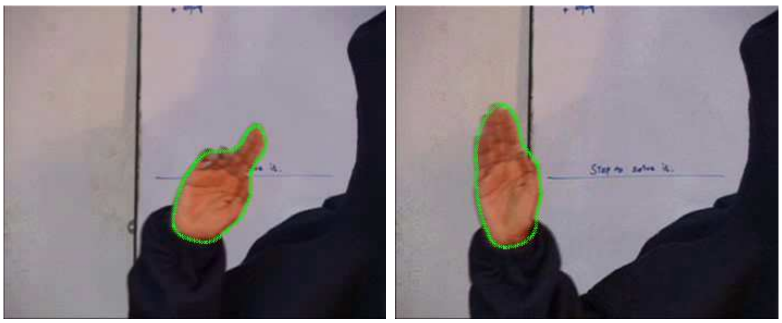

FIGURE 13. Tracking results for the same sequence as in Fig. 12 using a nontrivial tangential redistribution.

where $K = L(\Gamma) - \int_\Gamma \kappa\beta \, \mathrm{d}s$ we are able to significantly improve the results of tracking boundaries in moving images. If we compare tracking results in Fig. 13 and those from Fig. 12 we can conclude that the presence of a nontrivial suitably chosen tangential velocity α significantly improved tracking results.

Bibliography

[AG89] S. Angenent and M. E. Gurtin, *Multiphase thermomechanics with interfacial structure. II: Evolution of an isothermal interface.*, Arch. Ration. Mech. Anal. **108** (1989), no. 4, 323–391.

[AG94] S. Angenent and M. Gurtin, *Anisotropic motion of a phase interface. Well-posedness of the initial value problem and qualitative properties of the interface.*, J. Reine Angew. Math. **446** (1994), 1–47.

[AG96] ———, *General contact-angle conditions with and without kinetics*, Quart. Appl. Math. **54** (1996), no. 3, 557–569.

[AL86] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions.*, J. Differ. Geom. **23** (1986), 175–196.

[Ang90a] S. Angenent, *Nonlinear analytic semiflows.*, Proc. R. Soc. Edinb., Sect. A **115** (1990), no. 1/2, 91–107.

[Ang90b] ———, *Parabolic equations for curves on surfaces. I: Curves with p-integrable curvature.*, Ann. Math. **132** (1990), no. 2, 451–483.

[AST98] S. Angenent, G. Sapiro, and A. Tannenbaum, *On the affine heat equation for non-convex curves*, J. Amer. Math. Soc. **11** (1998), no. 3, 601–634.

[Bai80] J.-B. Baillon, *Caractère borné de certains générateurs de semi-groupes linéaires dans les espaces de Banach*, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 16, A757–A760.

[Ben01] M. Beneš, *Mathematical and computational aspects of solidification of pure substances.*, Acta Math. Univ. Comen., New Ser. **70** (2001), no. 1, 123–151.

[BM98] M. Beneš and K. Mikula, *Simulation of anisotropic motion by mean curvature–comparison of phase field and sharp interface approaches*, Acta Math. Univ. Comenian. (N.S.) **67** (1998), no. 1, 17–42.

[Bra78] K. Brakke, *The motion of a surface by its mean curvature.*, Princeton, New Jersey: Princeton University Press. Tokyo: University of Tokyo Press. 240 p., 1978.

[BS91] G. Barles and P. Souganidis, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic Anal. **4** (1991), no. 3, 271–283.

[Cag90] G. Caginalp, *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math. **44** (1990), no. 1, 77–94.

[Cao03] F. Cao, *Geometric curve evolution and image processing*, Lecture Notes in Mathematics, vol. 1805, Springer-Verlag, Berlin, 2003.

[CCCD93] V. Caselles, F. Catté, T. Coll, and F. Dibos, *A geometric model for active contours in image processing*, Numer. Math. **66** (1993), no. 1, 1–31.

[CGG91] Y. Chen, Y. Giga, and S. Goto, *Remarks on viscosity solutions for evolution equations*, Proc. Japan Acad. Ser. A Math. Sci. **67** (1991), no. 10, 323–328.

[CKS97] V. Caselles, R. Kimmel, and G. Sapiro, *Geodesic active contours*, International Journal of Computer Vision **22** (1997), 61–79.

[CKSS97] V. Caselles, R. Kimmel, G. Sapiro, and C. Sbert, *Minimal surfaces: a geometric three-dimensional segmentation approach*, Numer. Math. **77** (1997), no. 4, 423–451.

[CL83] M. Crandall and P.-L. Lions, *Viscosity solutions of hamilton–jacobi equations*, Trans. Amer. Math. Soc. **227** (1983), 1–42.

[Dec97] K. Deckelnick, *Weak solutions of the curve shortening flow.*, Calc. Var. Partial Differ. Equ. **5** (1997), no. 6, 489–510.

[DeT83] D. DeTurck, *The Cauchy problem for Lorentz metrics with prescribed Ricci curvature*, Compositio Math. **48** (1983), no. 3, 327–349.

[DPG75] G. Da Prato and P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles*, J. Math. Pures Appl. (9) **54** (1975), no. 3, 305–387.

[DPG79] ———, *Equations d'évolution abstraites non linéaires de type parabolique*, Ann. Mat. Pura Appl. (4) **120** (1979), 329–396.

[Dzi91] G. Dziuk, *An algorithm for evolutionary surfaces.*, Numer. Math. **58** (1991), no. 6, 603–611.

[Dzi94] ———, *Convergence of a semi-discrete scheme for the curve shortening flow.*, Math. Models Methods Appl. Sci. **4** (1994), no. 4, 589–606.

[Dzi99] ———, *Discrete anisotropic curve shortening flow.*, SIAM J. Numer. Anal. **36** (1999), no. 6, 1808–1830.

[EG87] C. Epstein and M. Gage, *The curve shortening flow*, Wave motion: theory, modelling, and computation (Berkeley, Calif., 1986), Math. Sci. Res. Inst. Publ., vol. 7, Springer, New York, 1987, pp. 15–59.

[EPS96] Ch. Elliott, M. Paolini, and R. Schätzle, *Interface estimates for the fully anisotropic Allen-Cahn equation and anisotropic mean-curvature flow*, Math. Models Methods Appl. Sci. **6** (1996), no. 8, 1103–1118.

[ES91] L. Evans and J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom. **33** (1991), no. 3, 635–681.

[FS93] W. Fleming and H. Soner, *Controlled Markov processes and viscosity solutions*, Applications of Mathematics (New York), vol. 25, Springer-Verlag, New York, 1993.

[GH86] M. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves.*, J. Differ. Geom. **23** (1986), 69–96.

[Gir95] P. Girão, *Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature*, SIAM J. Numer. Anal. **32** (1995), no. 3, 886–899.

[GK94] P. Girão and R. Kohn, *Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature*, Numer. Math. **67** (1994), no. 1, 41–70.

[Gra87] M. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987), 285–314.

[Gur93] M. Gurtin, *Thermomechanics of evolving phase boundaries in the plane*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993.

[Hen81] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.

[HKS98] T. Y. Hou, I. Klapper, and H. Si, *Removing the stiffness of curvature in computing 3-D filaments.*, J. Comput. Phys. **143** (1998), no. 2, 628–664.

[HLS94] T. Y. Hou, J. S. Lowengrub, and M. J. Shelley, *Removing the stiffness from interfacial flows with surface tension.*, J. Comput. Phys. **114** (1994), no. 2, 312–338.

[HMS98] A. Handlovičová, K. Mikula, and A. Sarti, *Numerical solution of parabolic equations related to level set formulation of mean curvature flow*, Comput. Visualization Sci. **1** (1998), 179–182.

[Hui90] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow.*, J. Differ. Geom. **31** (1990), no. 1, 285–299.

[IS95] H. Ishii and P. Souganidis, *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor*, Tohoku Math. J. (2) **47** (1995), no. 2, 227–250.

[Jen88] R. Jensen, *Viscosity solutions of hamilton–jacobi equations*, Arch. Rat. Mech. Anal. **101** (1988), 1–27.

[Kim94] M. Kimura, *Accurate numerical scheme for the flow by curvature*, Appl. Math. Lett. **7** (1994), no. 1, 69–73.

[Kim97] ———, *Numerical analysis of moving boundary problems using the boundary tracking method*, Japan J. Indust. Appl. Math. **14** (1997), no. 3, 373–398.

[KKO+96] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, and A. Yezzi, *Conformal curvature flows: from phase transitions to active vision*, Arch. Rational Mech. Anal. **134** (1996), no. 3, 275–301.

[KM95] J. Kačur and K. Mikula, *Solution of nonlinear diffusion appearing in image smoothing and edge detection*, Appl. Numer. Math. **17** (1995), no. 1, 47–59.

[KWT87] M. Kass, A. Witkin, and D. Terzopulos, *Snakes: active contour models*, International Journal of Computer Vision **1** (1987), 321–331.

[Lun82] A. Lunardi, *Analyticity of the maximal solution of an abstract nonlinear parabolic equation*, Nonlinear Anal. **6** (1982), no. 5, 503–521.

[Mik97] K. Mikula, *Solution of nonlinear curvature driven evolution of plane convex curves.*, Appl. Numer. Math. **23** (1997), no. 3, 347–360.

[MK96] K. Mikula and J. Kačur, *Evolution of convex plane curves describing anisotropic motions of phase interfaces.*, SIAM J. Sci. Comput. **17** (1996), no. 6, 1302–1327.

[Moi98] L. Moissan, *Affine plane curve evolution: A fully consistent scheme*, IEEE Trans. Image Process. **7** (1998), 411–420.

[MS99] K. Mikula and D. Ševčovič, *Solution of nonlinearly curvature driven evolution of plane curves.*, Appl. Numer. Math. **31** (1999), no. 2, 191–207.

[MS01] ———, *Evolution of plane curves driven by a nonlinear function of curvature and anisotropy*, SIAM Journal on Applied Mathematics **61** (2001), no. 5, 1473–1501.

[MS04a] ———, *A direct method for solving an anisotropic mean curvature flow of plane curves with an external force.*, Math. Methods Appl. Sci. **27** (2004), no. 13, 1545–1565.

[MS04b] ———, *Computational and qualitative aspects of evolution of curves driven by curvature and external force*, Comput. and Vis. in Science **6** (2004), no. 4, 211–225.

[MS06] ———, *Evolution of curves on a surface driven by the geodesic curvature and external force.*, Appl. Anal. **85** (2006), no. 4, 345–362.

[MSV95] R. Malladi, J. Sethian, and B. Vemuri, *Shape modeling with front propagation: a level set approach*, IEEE Trans. Pattern Anal. Mach. Intell. **17** (1995), 158–174.

[NPV93] R. Nochetto, M. Paolini, and C. Verdi, *Sharp error analysis for curvature dependent evolving fronts*, Math. Models Methods Appl. Sci. **3** (1993), no. 6, 711–723.

[OF03] S. Osher and R. Fedkiw, *Level set methods and dynamic implicit surfaces*, Applied Mathematical Sciences, vol. 153, Springer-Verlag, New York, 2003.

[OS88] S. Osher and J. Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys. **79** (1988), no. 1, 12–49.

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[Sap01] G. Sapiro, *Geometric partial differential equations and image analysis*, Cambridge University Press, Cambridge, 2001.

[SCDR07] V. Srikrishnan, S. Chaudhuri, and S. Dutta Roy, *Stabilisation of active contours using tangential evolution: An application to tracking*, Proceedings of International Conference on Computing: Theory and Applications (ICCTA'07) (2007), 660–664.

[SCDRS07] V. Srikrishnan, S. Chaudhuri, S. Dutta Roy, and Ševčovič, *On stabilisation of parametric active contours*, Proc. of IEEE Conference on Computer Vision and Pattern Recognition, to appear, 2007.

[Set90] J. A. Sethian, *Numerical algorithms for propagating interfaces: Hamilton-Jacobi equations and conservation laws*, J. Differential Geom. **31** (1990), no. 1, 131–161.

[Set96] J. Sethian, *Level set methods*, Cambridge Monographs on Applied and Computational Mathematics, vol. 3, Cambridge University Press, Cambridge, 1996, Evolving interfaces in geometry, fluid mechanics, computer vision, and materials science.

[Set98] ———, *Adaptive fast marching and level set methods for propagating interfaces*, Acta Math. Univ. Comenian. (N.S.) **67** (1998), no. 1, 3–15.

[SMS99] A. Sarti, K. Mikula, and F. Sgallari, *Nonlinear multiscale analysis of three-dimensional echocardiographi sequences*, IEEE Trans. on Medical Imaging **18** (1999), 453–466.

[ST94] G. Sapiro and A. Tannenbaum, *On affine plane curve evolution*, J. Funct. Anal. **119** (1994), no. 1, 79–120.

[UY00] T. Ushijima and S. Yazaki, *Convergence of a crystalline algorithm for the motion of a closed convex curve by a power of curvature* $V = K^\alpha$, SIAM J. Numer. Anal. **37** (2000), no. 2, 500–522.