Introduction to 3-D finite element computation

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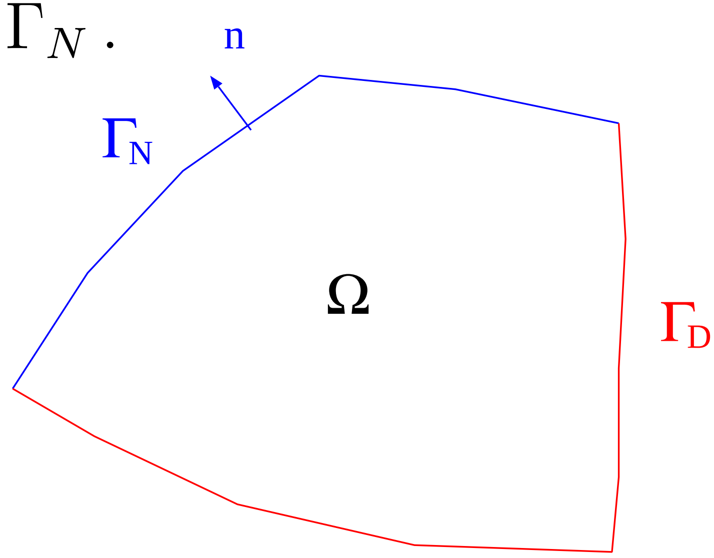
Global stiffness matrix form local element stiffness matrices and sparse matrix storage formats.

- Introduction
- finite element basis
  - piecewise linear polynomial
  - mesh subdivision
  - element-node table
- finite element equation
- stiffness matrix
  - local element-stiffness matrix
  - area coordinates
- storing sparse matrix

Navier equations :

$$\left\{\begin{array}{rcl} -\displaystyle\sum_{1\le j\le 3}\partial_j[\sigma(u)]_{i\,j} & = & [f]_i \quad (i=1,2,3)\ \text{in}\ \Omega\,, \\ u & = & g\ \text{ on } \Gamma_D\,, \\ \sigma(u)n & = & h\ \text{ on } \Gamma_N\,. \end{array}\right.$$



$\Omega\subset\mathbb{R}^3$: bounded polyhedral domain,

$\Gamma_D,\ \Gamma_N\subset\partial\Omega$,

Γ_D: closed, $\Gamma_D\cup\Gamma_N=\partial\Omega$.

stress tensor: $[\sigma(u)]_{i\,j}=\lambda(\sum_{1\le k\le 3}[\epsilon(u)]_{k\,k})\delta_{i\,j}+2\mu[\epsilon(u)]_{i\,j}$

Lamé constants: $\lambda>0$, $\mu>0$

strain rate tensor: $[\epsilon(u)]_{i\,j}=\dfrac{1}{2}(\partial_i[u]_j+\partial_j[u]_i)$

Weak formulation of the Navier equations

bilinear form and functional

$$a(u,v\,;\,\Omega) = 2\mu \int_\Omega \sum_{1\le i,j\le 3} [\epsilon(u)]_{i\,j}[\epsilon(v)]_{i\,j}\,dx + \lambda \int_\Omega \text{div}u\,\text{div}v\,dx,$$

$$l(v\,;\,\Omega) = \int_\Omega f\cdot v\,dx + \int_{\Gamma_N} h\cdot v\,ds$$

subset and subspace:

$$V(g) := \{v \in H^1(\Omega)^3\,;\, v|_{\Gamma_D} = g\}, \quad V := V(0)\,.$$

weak formulation:

Find $u \in V(g)$ such that

$$a(u,v) = l(v) \qquad \forall v \in V$$

discritization of the weak formulation $\rightarrow$ FEM

$a(\,\cdot\,,\,\cdot\,)$ is coercive on V .

$\exists\alpha > 0 \;\; \forall v \in V \;\;\; a(v,v) \geq \alpha ||v||_1^2$.

$||\cdot||_1$: $H^1(\Omega)^3$-norm.

Korn's inequality:

$\exists c > 0 \;\; \forall v \in H^1(\Omega)^3$

$$\int_\Omega \sum_{1\leq i,j\leq 3} [\epsilon(v)]_{i\,j}[\epsilon(v)]_{i\,j}\,dx + ||v||_0^3 \geq c||v||_1^2 \,.$$

[Duvaut-Lions 1972]

Finite element basis (1/3)

mesh subdivision

$$\mathcal{T}_h := \{K_l \ (l \in \Lambda_E) \, ; \ \bigcup_{l \in \Lambda_E} K_l = \bar{\Omega}\}$$

$\Lambda_E = \{1, 2, \ldots, N_E\}$: index set of elements

$\Lambda_G = \{1, 2, \ldots, N_G\}$: index set of nodes

piecewise polynomial of degree p

$$\mathcal{L}_p^1(\Omega) = \{v \in C_0(\bar{\Omega}) \, ; \ v|_K \in \mathcal{P}_p \, ; \ K \in \mathcal{T}_h\} \subset H^1(\Omega) \cap C(\bar{\Omega})$$

$\mathcal{P}_p$: polynomials of degree p

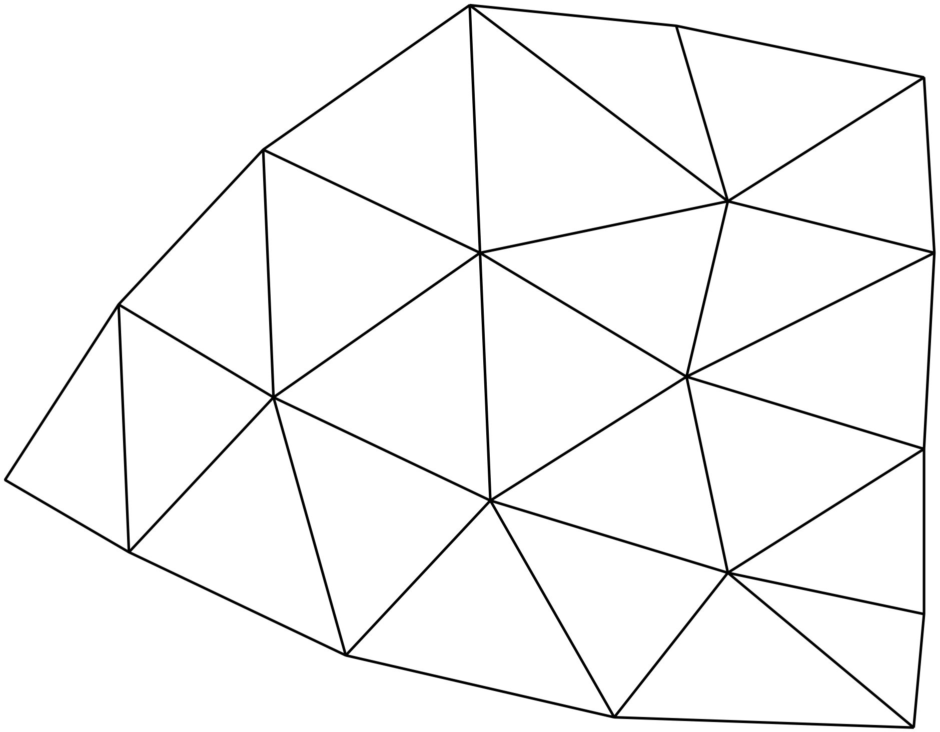
base function $\psi_\mu \in X_h := \mathcal{L}_k^1(\Omega)$

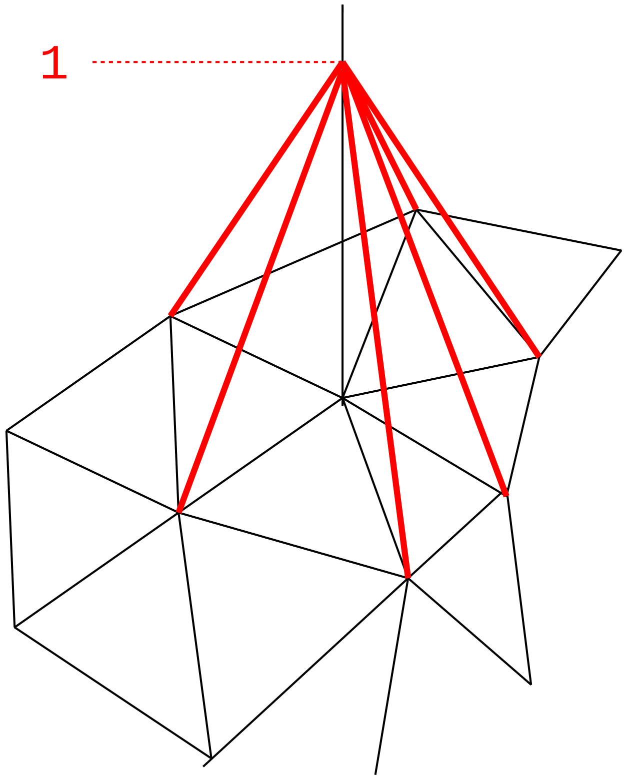
$$\psi_\mu(P_\nu) = \delta_{\mu\,\nu} \quad \forall \mu, \nu \in \Lambda_G$$

$$\text{span}[\{\psi_\mu\}_{\mu \in \Lambda_G}] = X_h$$

Finite element basis (2/3)

2-D case





mesh subdivision with triangles basis

Finite element basis (3/3)

$\Lambda_K = \{\tau^K(1), \tau^K(2), \dots, \tau^K(m)\}$: index set of node in K.

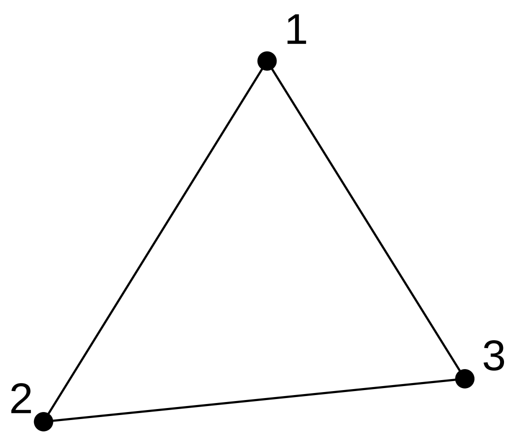
$$\varphi_\mu(x)|_K = \varphi^{(i)}(x) \quad x \in K,\ \tau^K(i) = \mu\ (i = 1, 2, \dots, m)$$

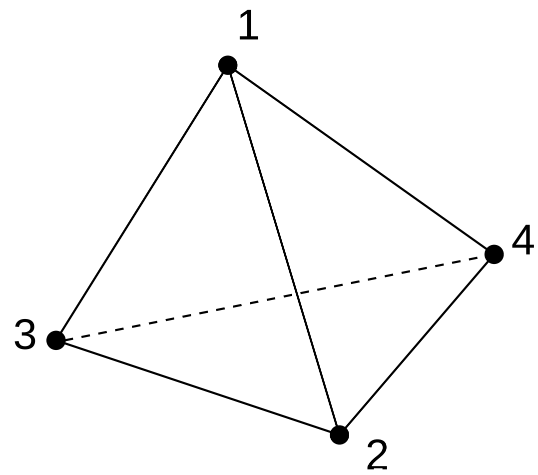
$\varphi^{(i)}(x)$: polynomial of degree k in K.

dim. d	order k	# of nodes m	polynomial
2	1	3	$a_0^{(i)} + a_1^{(i)} x_1 + a_2^{(i)} x_2$
2	2	6	$a_0^{(i)} + a_1^{(i)} x_1 + a_2^{(i)} x_2 +$ $a_3^{(i)} x_1^2 + a_4^{(i)} x_2^2 + a_5^{(i)} x_1 x_2$
3	1	4	$a_0^{(i)} + a_1^{(i)} x_1 + a_2^{(i)} x_2 + a_3^{(i)} x_3$
3	2	10	$a_0^{(i)} + a_1^{(i)} x_1 + a_2^{(i)} x_2 + a_3^{(i)} x_3 +$ $a_4^{(i)} x_1^2 + a_5^{(i)} x_2^2 + a_6^{(i)} x_3^2 +$ $a_7^{(i)} x_1 x_2 + a_8^{(i)} x_2 x_3 + a_9^{(i)} x_3 x_1$

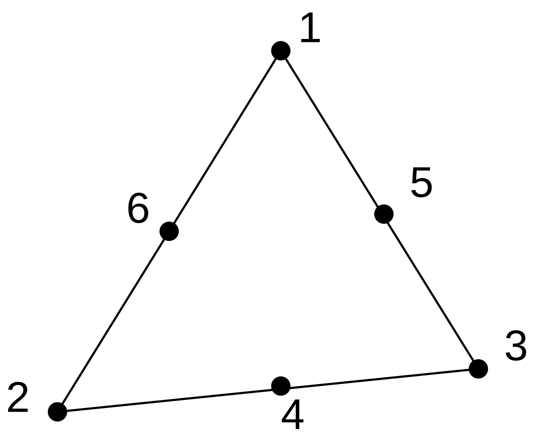
P1 element and P2 element

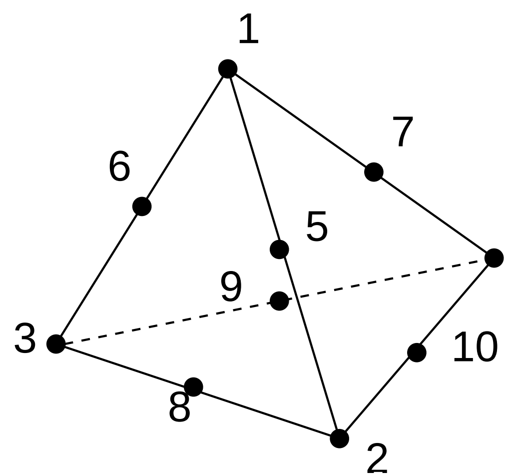
P1 element : $\{\text{node}\} = \{\text{vertex}\}$





P2 element : $\{\text{node}\} = \{\text{vertex}\} \cup \{\text{middle point of edge}\}$





Vector-valued basis

vector-valued finite element space: $Y_h := \mathcal{L}_p^1(\Omega)^3$

$$\varphi_\alpha = \psi_{\alpha_0} \mathbf{e}_{\alpha_1} \quad (\alpha = 3\alpha_0 + \alpha_1,\ \alpha_0 \in \Lambda_G, \alpha_1 \in \{1,2,3\}\ )$$

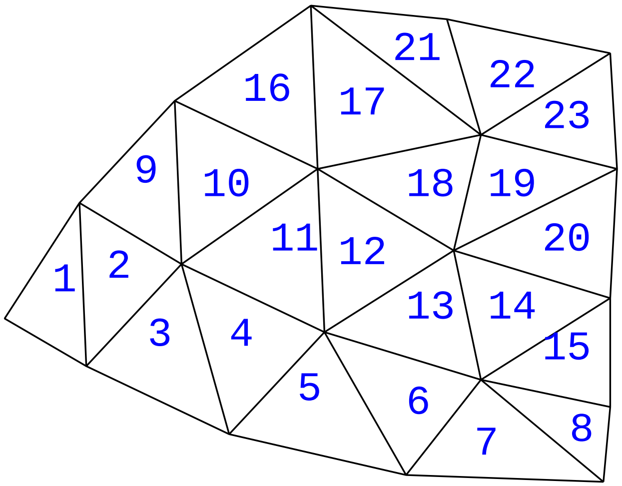
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$: 3-D normal basis

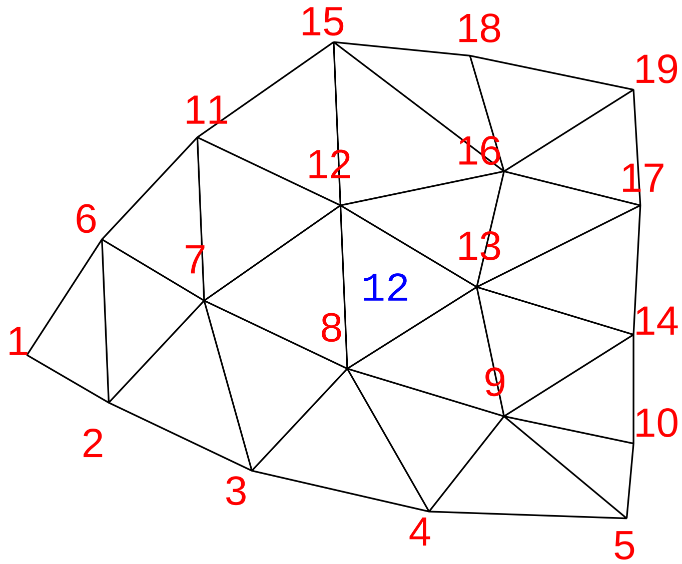
$$[\,\varphi_\alpha(P_{\beta_0})\,]_{\beta_1} = \delta_{\alpha\,\beta} \quad (\,\beta = 3\beta_0 + \beta_1\,)$$

index of vector-valued finite element space

$$\Lambda_Y = \{\alpha\,;\,\alpha = 3\alpha_0 + \alpha_1, \alpha_0 \in \Lambda_G, \alpha_1 \in \{1,2,3\}\}$$

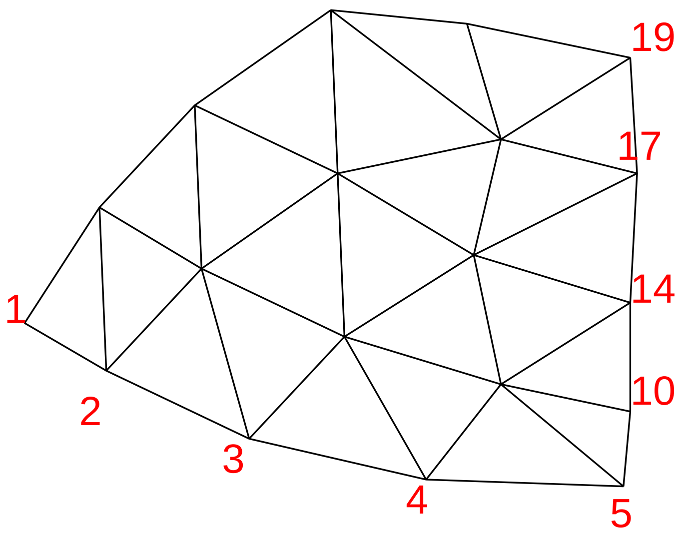
$N_Y := \#\Lambda_Y = 3N_G$

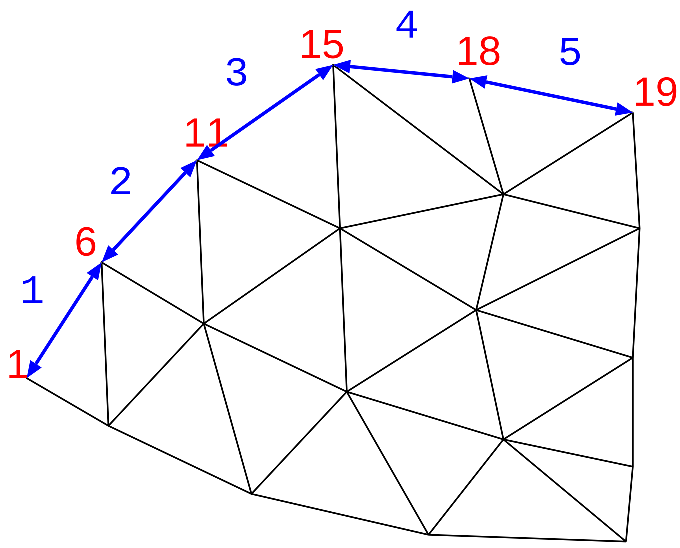




index of element	index of node		
l	$\{\tau^{K_l}(1),$	$\tau^{K_l}(2),$	$\tau^{K_l}(3)\}$
$\vdots$		$\vdots$	
11	7	8	12
12	8	13	12
13	8	9	13
14	9	14	13
$\vdots$		$\vdots$	

Dirichlet boundary and Neumann boundary





$$\Lambda_G \supset \Lambda_D := \{1, 2, 3, 4, 5, \\ 10, 14, 17, 19\}$$

$$N_D := \#\Lambda_D$$

idx of elmt	idx of nd	
1	1	6
2	6	11
3	11	15
4	15	18
5	18	19

input data

coordinates of nodes : $\{P_\mu(x_1^{(\mu)}, x_2^{(\mu)}, x_3^{(\mu)})\}_{\mu\in\Lambda_G}$

element-node table : $\{l, \{\tau^{K_l}(1), \tau^{K_l}(2), \tau^{K_l}(3), \tau^{K_l}(4)\}\}_{l\in\Lambda_E}$

node for boundary conditions : Λ_D, $\ldots$

$\Downarrow$

program code

generation of stiffness matrix

generation of load vector

$\Downarrow$

linear / nonlinear solver

$\Downarrow$

output data

visualization

Finite element equation (1/2)

bilinear form and functional

$$a(u, v\,;\,\Omega) = 2\mu \int_\Omega \sum_{1\le i,j\le 3} [\epsilon(u)]_{i\,j} [\epsilon(v)]_{i\,j}\, dx + \lambda \int_\Omega \text{div} u \text{div} v\, dx,$$

$$l(v\,;\,\Omega) = \int_\Omega f\cdot v\, dx + \int_{\Gamma_N} h\cdot v\, ds$$

affine space and subspace:

$$V_h(g) := \{v_h \in Y_h\,;\, v_h(P_\alpha) = g(P_\alpha) \;\; \forall \alpha \in \Lambda_D\}$$

$$V_h := V_h(0) = \{v_h \in Y_h\,;\, v_h(P_\alpha) = 0 \;\; \forall \alpha \in \Lambda_D\}$$

finite element equation:

Find $u_h \in V_h(g)$ such that

$$a(u_h, v_h) = l(v_h) \qquad \forall v_h \in V_h$$

Finite element equation (2/2)

$\{u_\beta\}_{\beta\in\Lambda_Y}$: data on nodes

$\{\varphi_\beta\}_{\beta\in\Lambda_Y}$: basis

$$Y_h \ni u_h = \sum_{\beta\in\Lambda_Y} u_\beta \varphi_\beta$$

Find $\{u_\beta\}_{\beta\in\Lambda_Y}$ such that

$$\sum_{\beta\in\Lambda_Y} a(\varphi_\beta, \varphi_\alpha) u_\beta = l(\varphi_\alpha) \qquad \forall\alpha \in \Lambda_Y \setminus \Lambda_D$$

$$u_\beta = g(P_\beta) \qquad (\beta \in \Lambda_D)$$

linear equation with $(N_Y - N_D)$ unknowns :

Find $\{u_\beta\}_{\beta\in\Lambda_Y\setminus\Lambda_D}$ such that

$$\sum_{\beta\in\Lambda_Y\setminus\Lambda_D} a(\varphi_\beta, \varphi_\alpha) u_\beta = l(\varphi_\alpha) - \sum_{\beta\in\Lambda_D} a(\varphi_\beta, \varphi_\alpha) g_\beta \qquad \forall\alpha \in \Lambda_Y \setminus \Lambda_D$$

Global stiffness matrix from local element-stiffness matrices

stiffness matrix A ($N_Y \times N_Y$)

$$A_{\alpha\,\beta} = a(\varphi_\beta, \varphi_\alpha\,;\,\Omega) \qquad \alpha\,,\beta \in \Lambda_Y$$

$$a(\varphi_\alpha, \varphi_\beta\,;\,\Omega) = \sum_{l\in\Lambda_E} a(\varphi_\beta, \varphi_\alpha\,;\,K_l)$$

index of nodes $\alpha, \beta \in \Lambda_Y$, $\alpha = 3\alpha_0 + \alpha_1$, $\beta = 3\beta_0 + \beta_1$.

$$P_{\alpha_0}, P_{\beta_0} \in K_l$$
$$\Updownarrow$$

index of nodes in an element $\alpha_0, \beta_0 \in \{1, 2, 3, 4\}$

element-stiffness matrix (12×12)

$$a(\varphi^{(\beta)}, \varphi^{(\alpha)}\,;\,K_l)$$

K_l : tetrahedral element $\{1,2,3,4\}$: index of nodes in K_l

$\{P_j(x_1^{(j)},x_2^{(j)},x_3^{(j)})\}_{j=1,2,3,4}$: vertex

$\psi^{(i)}(x) = a_0^{(i)} + a_1^{(i)}x_1 + a_2^{(i)}x_2 + a_3^{(i)}x_3$: basis

$$\psi^{(i)}(P_j) = \delta_{i\,j}\ (1 \leq i,j \leq 4)$$

linear equation:

$$\begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_3^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_3^{(2)} \\ 1 & x_1^{(3)} & x_2^{(3)} & x_3^{(3)} \\ 1 & x_1^{(4)} & x_2^{(4)} & x_3^{(4)} \end{bmatrix} \begin{bmatrix} a_0^{(i)} \\ a_1^{(i)} \\ a_2^{(i)} \\ a_3^{(i)} \end{bmatrix} = \begin{bmatrix} \delta_{i\,1} \\ \delta_{i\,2} \\ \delta_{i\,3} \\ \delta_{i\,4} \end{bmatrix}.$$

$\Rightarrow a_0^{(i)}$, $a_1^{(i)}$, $a_2^{(i)}$, $a_3^{(i)}$ Cramer's formula

determinant $=$ vol(K_l) $\times 6$.

Local element stiffness matrix with P1 element

$$\frac{\partial}{\partial x_k}\psi^{(i)} = a_k^{(i)} \ (k = 1, 2, 3)$$

$$\int_{K_l} \mathrm{div}\varphi^{(\beta)}\mathrm{div}\varphi^{(\alpha)}\, dx = a_{\beta_1}^{(\beta_0)} a_{\alpha_1}^{(\alpha_0)} \int_{K_l} dx\,.$$

$$\int_\Omega \sum_{1\le i,j\le 3} [\epsilon(\varphi^{(\beta)})]_{i\,j}[\epsilon(\varphi^{(\alpha)})]_{i\,j}\, dx =$$

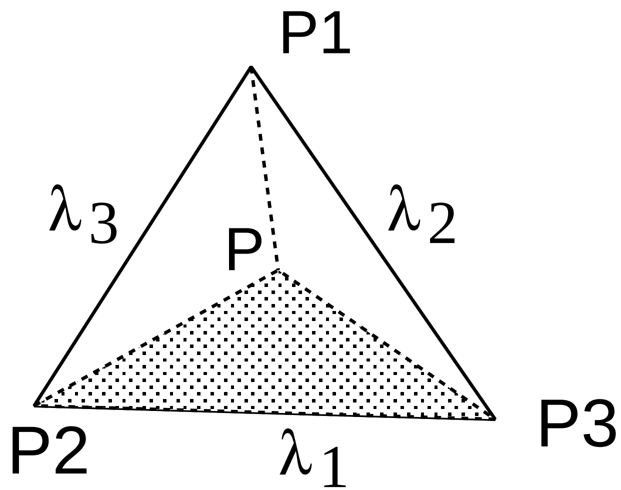
$$\frac{1}{2}(a_{\beta_1}^{(\alpha_0)} a_{\alpha_1}^{(\beta_0)} + \delta_{\alpha\,\beta} \sum_{1\le i\le 3} a_i^{(\alpha_0)} a_i^{(\beta_0)}) \int_{K_l} dx\,.$$

element mass matrix (12×12)

$$(\varphi^{(\beta)}, \varphi^{(\alpha)}\,;\, K_l) = \int_{K_l} \varphi^{(\beta)} \cdot \varphi^{(\alpha)}\, dx\,.$$

P_1, P_2, P_3 : vertices of triangle K

$\triangle(P\,P_j\,P_k)$ =area of triangle $P\,P_j\,P_k$,



$$\lambda_i(x) = \frac{\triangle P\,P_j\,P_k}{\triangle K}$$

$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

$\lambda_i(x)$: polynomial of degree 1 with $x = (x_1, x_2)$.

$$\lambda_i(x_1^{(j)}, x_2^{(j)}) = \delta_{i\,j}\,, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1\,.$$

$$\lambda_i = \varphi^{(i)} \qquad (i = 1, 2, 3)$$

integration on K

$$\int_K \lambda_1^k \lambda_1^l \lambda_2^m dx_1 dx_2 = \frac{2!\,k!\,l!\,m!}{(3+k+l+m)!} \triangle K$$

Numerical quadrature in 2D

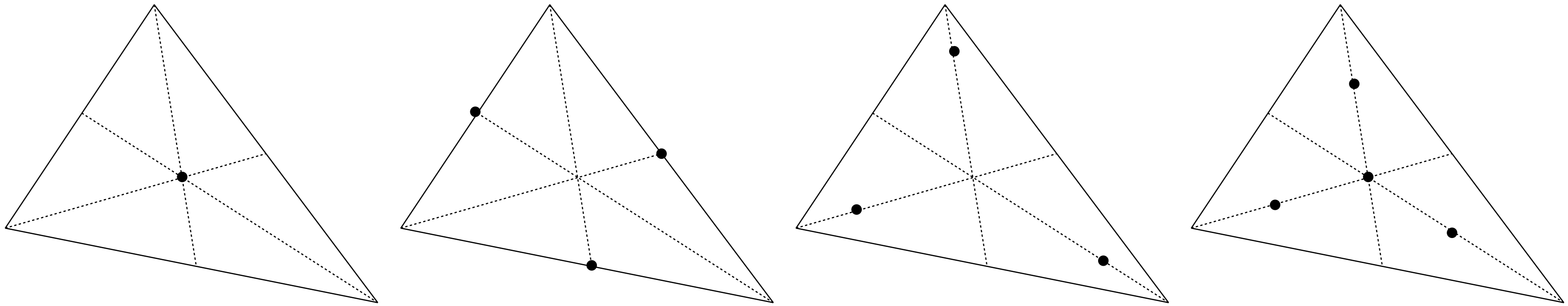
sampling point Q_i, weight w_i [Stroud 1971], [Zienkiewicz 2005]

$$\int_K f\,dx \sim \sum_{i=1}^{M} w_i f(Q_i)$$

numerical quadrature in triangle

order p	sampling M	weight w_i	area coordinates $\hat{Q}_i$
1	1	$\triangle K$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2	3	$\frac{1}{3}\triangle K$	(t, s, s) cyclic, $s = 1/2$, $t = 0$
2	3	$\frac{1}{3}\triangle K$	(t, s, s) cyclic, $s = 1/6$, $t = 2/3$
4	4	$-\frac{9}{16}\triangle K$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
		$\frac{25}{48}\triangle K$	(t, s, s) cyclic, $s = 1/5$, $t = 3/5$

integrand f : p-th order $\Rightarrow$ p-th order quadrature rule is required

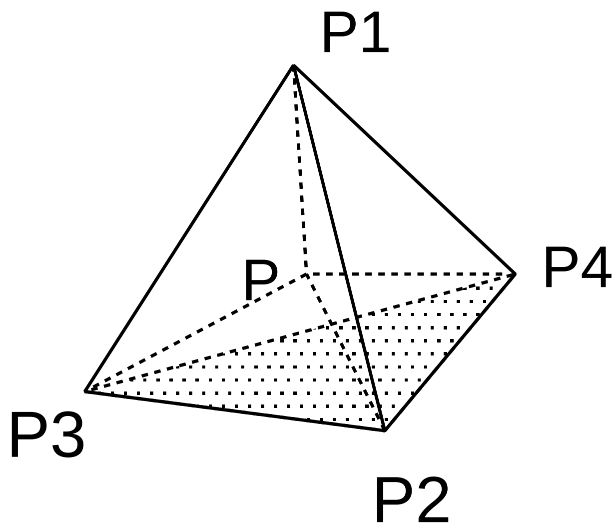


numerical quadrature in triangle

order	sampling	weight	area coordinates
p	M	w_i	$\hat{Q}_i$
1	1	$\triangle K$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2	3	$\frac{1}{3}\triangle K$	(t, s, s) cyclic, $s = 1/2$, $t = 0$
2	3	$\frac{1}{3}\triangle K$	(t, s, s) cyclic, $s = 1/6$, $t = 2/3$
4	4	$-\frac{9}{16}\triangle K$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
		$\frac{25}{48}\triangle K$	(t, s, s) cyclic, $s = 1/5$, $t = 3/5$

P_1, P_2, P_3, P_4 : vertices of tetrahedra K

Vol$(P\,P_j\,P_k\,P_l)$ =volume of tetrahedra $P\,P_j\,P_k\,P_l$,



$$\lambda_i(x) = \frac{\triangle P\,P_j\,P_k\,P_l}{\triangle K}$$

$(i,j,k,l) = (1,2,3,4), (2,3,1,4), \ldots$.

$\lambda_i(x)$: polynomial of degree 1 with $x = (x_1, x_2, x_3)$.

$$\lambda_i(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) = \delta_{i\,j}\,, \qquad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1\,.$$

$$\lambda_i = \varphi^{(i)} \qquad (i = 1,2,3,4)$$

integration on K

$$\int_K \lambda_1^k \lambda_1^l \lambda_2^m \lambda_3^n dx_1 dx_2 = \frac{3!\,k!\,l!\,m!\,n!}{(3+k+l+m+n)!} \triangle K$$

sampling point Q_i, weight w_i [Stroud 1971], [Zienkiewicz 2005]

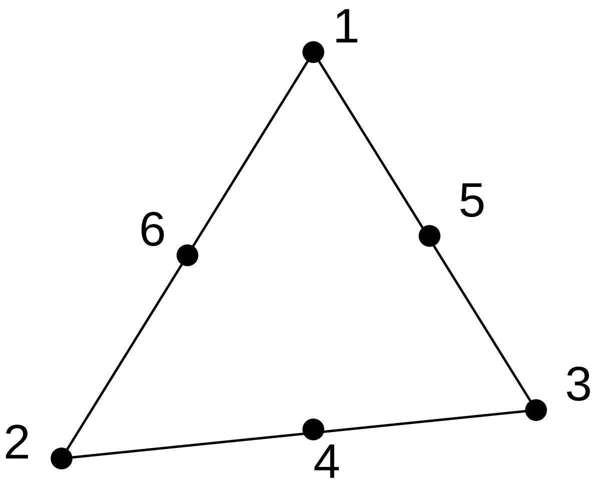
$$\int_K f dx \sim \sum_{i=1}^{M} w_i f(Q_i)$$

numerical quadrature in tetrahedra

order p	sampling M	weight w_i	volume coordinates $\hat{Q}_i$
1	1	$\triangle K$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
2	4	$\frac{1}{4} \triangle K$	(t, s, s, s) cyclic $s = (5 - \sqrt{5})/20$, $t = (5 + 3\sqrt{5})/20$
3	5	$-\frac{4}{5} \triangle K$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
		$\frac{9}{20} \triangle K$	(t, s, s, s) cyclic $s = 1/6$, $t = 1/2$

P2 element in 2D

P2 element and area coordinates



$$
\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{pmatrix}
=
\begin{pmatrix}
1 & & & & -1 & -1 \\
 & 1 & & -1 & & -1 \\
 & & 1 & -1 & -1 & \\
 & & & 4 & & \\
 & & & & 4 & \\
 & & & & & 4
\end{pmatrix}
\begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \\ \lambda_2\lambda_3 \\ \lambda_3\lambda_1 \\ \lambda_1\lambda_2 \end{pmatrix}
=
\begin{pmatrix} \lambda_1(2\lambda_1 - 1) \\ \lambda_2(2\lambda_2 - 1) \\ \lambda_3(2\lambda_3 - 1) \\ 4\lambda_2\lambda_3 \\ 4\lambda_3\lambda_1 \\ 4\lambda_1\lambda_2 \end{pmatrix}
$$

right-hand term of the finite element equation

$$l(\varphi_\alpha\,;\,\Omega) = \int_\Omega f \cdot \varphi_\alpha \, dx + \int_{\Gamma_N} h \cdot \varphi_\alpha \, ds \qquad \alpha \in \Lambda_Y$$

L^2-inner product of f and φ_μ:

1. interpolant of f

$$f \sim f_h = \Pi_h f = \sum_{\beta \in \Lambda_Y} f(P_\beta)\varphi_\beta$$

using mass matrix

$$\sum_{\beta \in \Lambda_Y} f_\beta \int_\Omega \varphi_\beta \cdot \varphi_\alpha \, dx = \sum_{\beta \in \Lambda_Y} f_\beta \sum_{l \in \Lambda_E} \int_{K_l} \varphi_\beta \cdot \varphi_\alpha \, dx \,.$$

2. using numerical quadrature

$$\int_\Omega f \varphi_\mu dx = \sum_{l \in \Lambda_E} \int_{K_l} f \cdot \varphi_\mu \, dx \sim \sum_{l \in \Lambda_E} \sum_{i=1}^{M} w_i f(Q_i) \cdot \varphi_\mu(Q_i) \,.$$

Load vector (2/2)

Neumann boundary conditions

$$\int_{\Gamma_N} h \cdot \varphi_\alpha \, ds = \sum_{l \in \Lambda_E} \int_{\Gamma_N \cap K_l} h \cdot \varphi_\alpha \, ds$$

$\Gamma_N \cap K_l$: surface triangle,

φ_μ : polynomial of degree p on the surface triangle.

$\Rightarrow$ quadrature for p-th order element in 2D.

right-hand term of the finite element equation :

$$l(\varphi_\alpha) - \sum_{\beta \in \Lambda_D} a(\varphi_\beta, \varphi_\alpha) g_\beta \qquad (\, \alpha \in \Lambda_Y \setminus \Lambda_D \,)$$

1. computation with index set of nodes for Dirichlet data: $\beta \in \Lambda_D$ and index set of test functions: $\alpha \in \Lambda_Y \setminus \Lambda_D$.

2. $N_Y \times N_Y$ stiffness matrix (defined on the whole index of nodes)

$$\tilde{g}_\beta = \begin{cases} 0 & \beta \notin \Lambda_D \\ g_\beta & \beta \in \Lambda_D \end{cases}$$

$$\sum_{\beta \in \Lambda_Y} a(\varphi_\beta, \varphi_\alpha) \tilde{g}_\beta = \sum_{\beta \in \Lambda_D} a(\varphi_\beta, \varphi_\alpha) g_\beta \qquad (\, \alpha \in \Lambda_Y \setminus \Lambda_D \,)$$

Calculation of left-hand term:

$$\sum_{\beta \in \Lambda_Y \setminus \Lambda_D} a(\varphi_\beta, \varphi_\alpha) u_\beta \quad (\, \alpha \in \Lambda_Y \setminus \Lambda_D \,)$$

product of stiffness matrix and vector $\rightarrow$ Krylov subspace methods

$$\tilde{u}_\beta = \begin{cases} u_\beta & \beta \notin \Lambda_D \\ 0 & \beta \in \Lambda_D \end{cases}$$

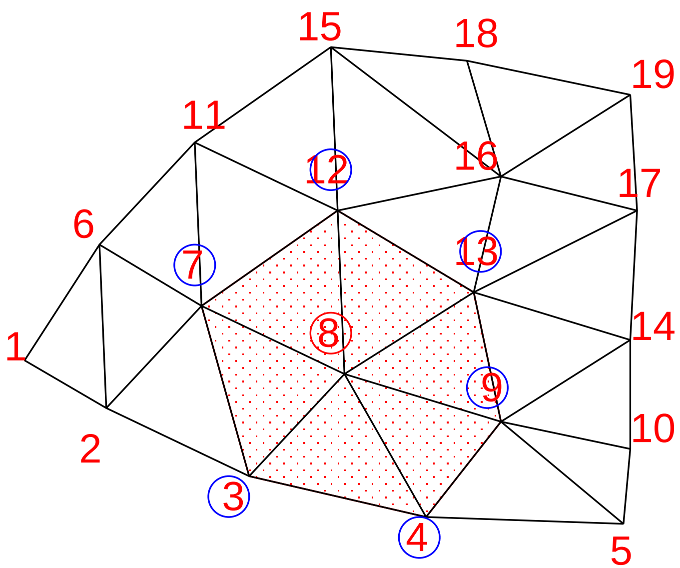
$N_Y \times N_Y$ stiffness matrix

$$\sum_{\beta \in \Lambda_Y} a(\varphi_\beta, \varphi_\alpha) \tilde{u}_\beta \quad (\, \alpha \in \Lambda_Y \supset \Lambda_Y \setminus \Lambda_D \,)$$

Storing sparse matrix (1/2)

stiffness/mass matrix : almost components are 0 and sparse

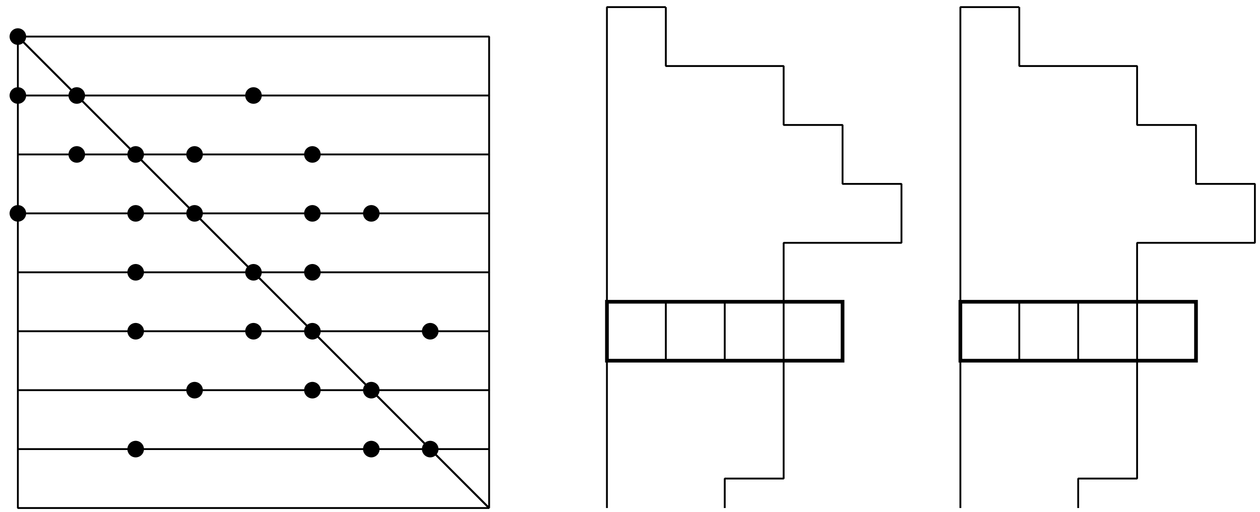
{# of nonzeros in the row} = {# of neighbor nodes to a node with the index of the row}



row	nonzero index of column						
⋮				⋮			
7	2	3	6	7	8	11	12
8	3	4	7	8	9	12	13
9	4	5	8	9	10	13	14
10	5	9	10	14			
⋮				⋮			

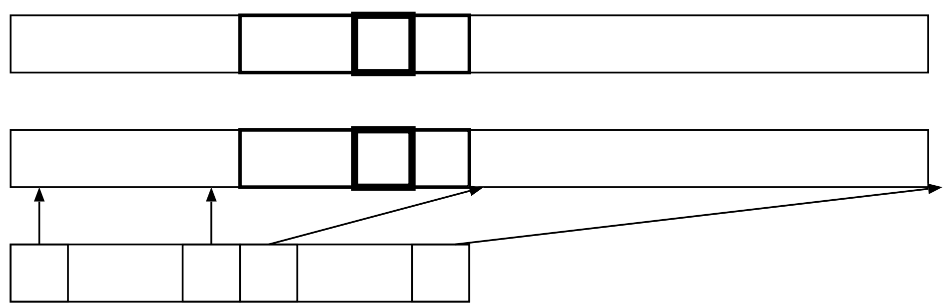
A : stiffness matrix with N_G-nodes, N_A : # of nonzeros

CRS (Compressed Row Storage) / CSR (Compressed Sparse Row) format [Barret et al. 1994],[Saad 2003]



A $v_A \in \mathbb{R}^{N_A}$, $c_A \in \mathbb{N}^{N_A}$, $r_A \in \mathbb{N}^{N_G+1}$

$$v_A[k] = A_{\mu\nu}$$
$$c_A[k] = \nu$$
$$r_A[\mu] \le k < r_A[\mu+1]$$



$r_A[1] = 1$, $r_A[\mu+1] \leftarrow ($ # of nonzeros up to μ-th row$)+1$.

Algorithm to generate stiffness matrix with CRS format

$\{\mathcal{P}[\mu]\}_{\mu\in\Lambda_G}$: work array to store neighbors of node P_μ.

initialization

$\mathcal{P}[\mu] = \emptyset \quad (\mu \in \Lambda_G)$

generating index set of neighbors

do $\;l = 1, 2, \ldots, N_E$

  do $i = 1, 2, \ldots, m$

    do $j = 1, 2, \ldots, m$

      $\mathcal{P}[\tau^{K_l}(i)] \leftarrow \mathcal{P}[\tau^{K_l}(i)] \cup \{\tau^{K_l}(j)\}$

generating index sets c_A and r_A

$r_A[1] = 1$

do $\;\nu = 1, 2, \ldots, N_p$

  $r_A[\nu+1] \leftarrow r_A[\nu] + \#\mathcal{P}[\nu]$

  $\{c_A[r_A[\nu]], \ldots, c_A[r_A[\nu+1]-1]\} \leftarrow \mathcal{P}[\nu]$

$N_A \leftarrow r_A[\nu+1]$

generating the whole stiffness matrix from element stiffness matrix

$v_A[k][(\alpha_1,\, \beta_1)] = 0 \quad (k = 1, 2, \ldots, N_A,\, 1 \le \alpha_1, \beta_1 \le 3))$

do $l = 1, 2, \ldots, N_e$

  do $i = 1, 2, \ldots, m$

    do $j = 1, 2, \ldots, m$

      find k $(r_A[\tau^{K_l}(i)] \le k < r_A[\tau^{K_l}(i)+1])$ satisfies $c_A[k] = \tau^{K_l}(j)$.

      do $\alpha_1 = 1, 2, 3, \beta_1 = 1, 2, 3$

        $v[k][(\alpha_1, \beta_1)] \leftarrow v[k][(\alpha_1, \beta_1)] + a(\varphi^{(3j+\beta_1)}, \varphi^{(3i+\alpha_1)}\,;\, K_l)$

Algorithm to generate index set of neighbors

```
typedef struct {
   int *idx;
   int n;
} sparse_matrix_index;
sparse_matrix_index 𝒫[N_G];
for (μ ∈ Λ_G){
  𝒫[μ].idx=(int *)malloc(sizeof(int)*M);
  𝒫[μ].n=0;
}
for (l ∈ Λ_E){
  for (i ∈ {1,2,...,m}){
    n=𝒫[τ^{K_l}(i)].n;
    for (j ∈ {1,2,...,m}){
      if (n>0 && n%M==0)
        resize(𝒫[μ ].idx, +M);
      find k ( 𝒫[τ^{K_l}(i)].idx[k] < τ^{K_l}(j) < 𝒫[τ^{K_l}(i)].idx[k+1]);
      𝒫[τ^{K_l}(i)].idx= {𝒫[τ^{K_l}(i)].idx[0],..., 𝒫[τ^{K_l}(i)].idx[k],
                          τ^{K_l}(j), 𝒫[τ^{K_l}(i)].idx[k+1],..., 𝒫[τ^{K_l}(i)].idx[n]};
      𝒫[τ^{K_l}(i)].n++;
    }
  }
}
```

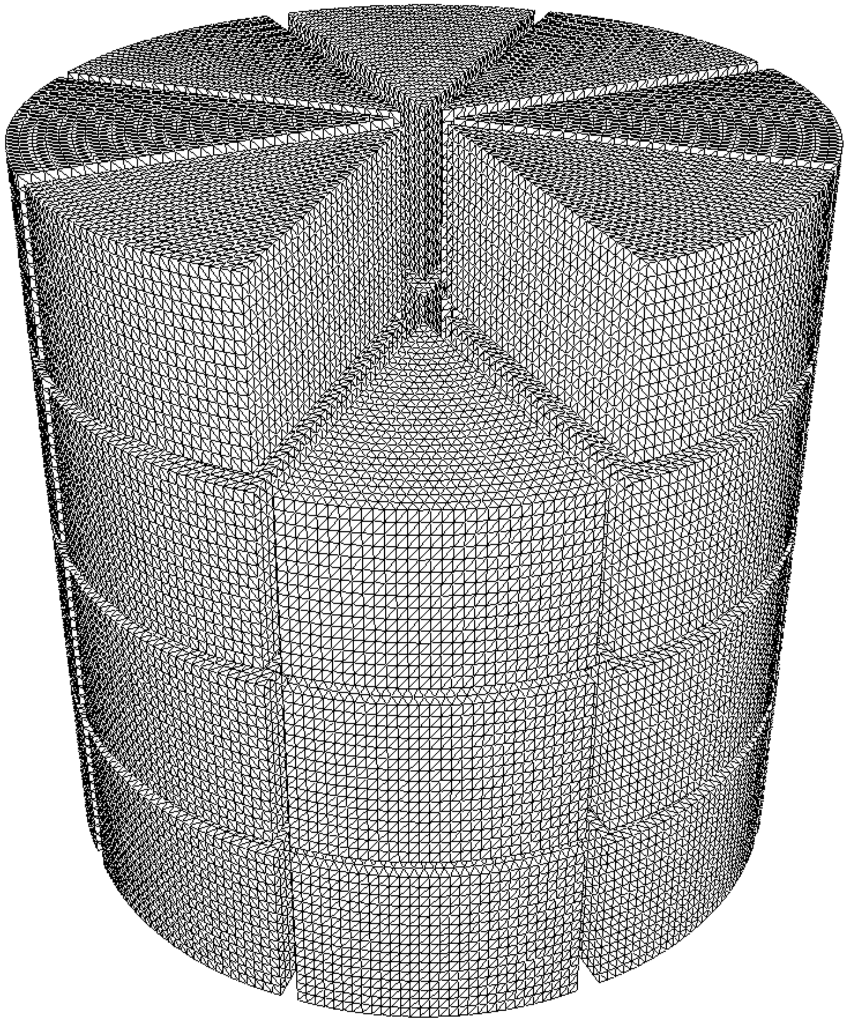
Remarks

Procedures to generate stiffness matrix for 2D and 3D are common.

Stiffness matrix is a large-scale sparse matrix in 3-D problem $\Rightarrow$ Krylov subspace methods are efficient to solve finite element equations

CRS format reduces required memory

An example of finite element mesh



n_G	n_E	n_X	n_Y
226,981	1,296,000	226,981	680,943

References

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