Analytical and numerical methods of financial-derivative pricing

D. Ševčovič

Comenius University, Bratislava

September 10, 2009

Seminars at Czech Technical University, Fall 2009

Outline

- Stochastic character of assets (stocks, indices, ...)
- Financial derivatives as tool for protecting volatile portfolios
- Stochastic differential calculus, Itō's lemma, Itō's integral
- Pricing European type of options - Black–Scholes model
- Transaction costs and Black –Scholes model
- Some exotic options – Asian, barrier, Russian options
- Modeling of stochastic interest rates
- Interest rate derivatives – bonds, swaps ...
- American type options – free boundary problems
- Numerical methods for pricing American type of options
- Conclusions

- Stochastic character of assets (stocks, indices, ...)
- Financial derivatives as tool for protecting volatile portfolios

Stochastic character of stock prices

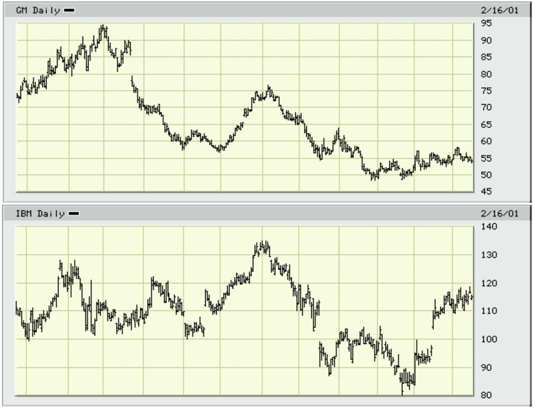


Figure: Time evolution of stock prices General Motors and IBM in 2001.

Stochastic character of stock prices

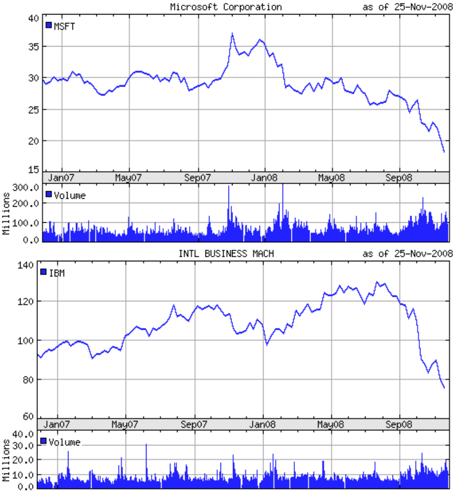


Figure: Time evolution of stock prices Microsoft and IBM in 2007, 2008. Volume of transaction.

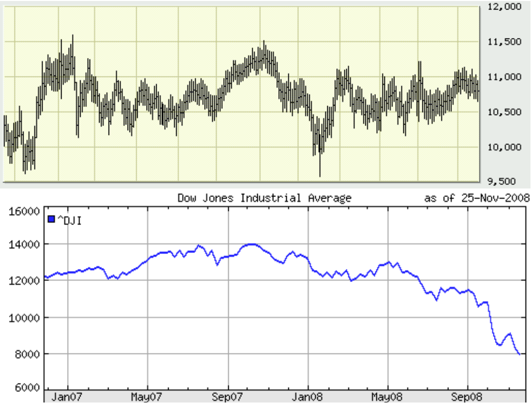


Figure: Time evolution of Dow–Jones index in precrisis periods 2000 and 2007-8.

Financial derivatives as a tool for protecting volatile portfolios

- Forward
  is an agreement between writer (issuer) and holder representing the right and at the same time obligation to purchase assets at the specified time of maturity of a forward

Pricing forwards is relatively simple once we know the interest rate r measuring the rate of the decrease of the value of money

$$V_f = E \exp(-rT)$$

where E is the contracted expiration value of a forward at expiration time T, V_f is the present value of a forward at the time when contract is signed

Financial derivatives as a tool for protecting volatile portfolios

- Option (call option)
  is an agreement between writer (issuer) and holder representing the right BUT NOT the obligation to purchase assets at the prescribed exercise price E at the specified time of maturity T in the future

Pricing option is more involved

$$V_c = \text{function of} \quad E, T, r, ..., ???$$

where E is the contracted expiration value of a forward at expiration time T, V_c is the present value of a call option at the time when contract is signed

Stochastic character of options

Microsoft Corporation (MSFT) At 9:41AM ET: **20.12** ↑ 0.13 (0.65%)

Options

View By Expiration: **Dec 08** | Jan 09 | Apr 09 | Jul 09 | Jan 10 | Jan 11

Options Expiring Fri, Dec 19, 2008

Calls							Strike Price	Puts						
Symbol	Last	Change	Bid	Ask	Volume	Open Int		Symbol	Last	Change	Bid	Ask	Volume	Open Int
MQFLE.X	**15.20**	0.00	15.10	15.20	42	34	**5.00**	MQFXE.X	N/A	0.00	N/A	N/A	0	0
MQFLB.X	**10.15**	0.00	10.10	10.20	74	2,541	**10.00**	MQFXB.X	**0.03**	0.00	0.02	0.04	97	3,473
MQFLM.X	**7.20**	0.00	7.15	7.25	95	187	**13.00**	MQFXM.X	**0.07**	0.00	0.05	0.07	459	2,994
MQFLN.X	**6.15**	0.00	6.15	6.25	55	211	**14.00**	MQFXN.X	**0.10**	0.00	0.07	0.10	204	2,147
MQFLC.X	**5.06**	↑ 0.11	5.20	5.30	11	1,348	**15.00**	MQFXC.X	**0.14**	0.00	0.13	0.14	5	8,183
MQFLO.X	**4.35**	0.00	4.25	4.35	263	368	**16.00**	MQFXO.X	**0.20**	↓ 0.02	0.19	0.21	2	337
MQFLQ.X	**3.40**	0.00	3.30	3.40	122	4,157	**17.00**	MQFXQ.X	**0.32**	↓ 0.02	0.33	0.34	11	8,395
MQFLS.X	**1.83**	↓ 0.05	1.89	1.92	36	7,567	**19.00**	MQFXS.X	**0.83**	↑ 0.06	0.77	0.80	169	31,116
MQFLD.X	**1.28**	↓ 0.02	1.27	1.29	56	8,886	**20.00**	MQFXD.X	**1.14**	↓ 0.06	1.13	1.16	109	23,562
MQFLU.X	**0.78**	↓ 0.09	0.75	0.78	105	72,937	**21.00**	MQFXU.X	**1.83**	↑ 0.23	1.65	1.68	1	72,472
MSQLN.X	**0.40**	↓ 0.04	0.41	0.43	350	16,913	**22.00**	MSQXN.X	**2.58**	↑ 0.23	2.30	2.36	3	4,495
MSQLQ.X	**0.21**	↓ 0.01	0.20	0.22	125	20,801	**23.00**	MSQXQ.X	**3.10**	0.00	3.05	3.15	30	3,840
MSQLD.X	**0.09**	↓ 0.02	0.09	0.11	92	12,207	**24.00**	MSQXD.X	**3.80**	0.00	3.95	4.05	167	3,871
MSQLE.X	**0.04**	↓ 0.02	0.04	0.05	165	14,193	**25.00**	MSQXE.X	**4.90**	0.00	4.85	4.95	157	2,075
MSQLR.X	**0.02**	0.00	0.02	0.03	161	9,359	**26.00**	MSQXR.X	**6.15**	0.00	5.85	5.95	210	1,795
MSQLS.X	**0.02**	0.00	N/A	0.03	224	3,643	**27.00**	MSQXS.X	**7.00**	0.00	6.85	6.95	45	1,156
MSQLT.X	**0.02**	0.00	N/A	0.02	59	2,938	**28.00**	MSQXT.X	**7.55**	0.00	7.80	7.95	24	874
MSQLF.X	**0.01**	0.00	N/A	0.02	10	1,330	**30.00**	MSQXF.X	**10.54**	0.00	9.85	10.00	26	124

Highlighted options are in-the-money.

Figure: Prices of call and put options with different exercise (strike) prices E for Microsoft from 26. 11. 2008.

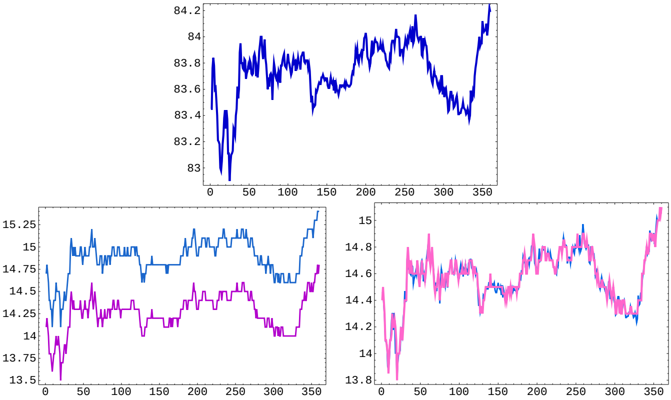


Figure: Top: Stock prices of IBM from 22. 5. 2002. Bottom: Bid and Ask prices of call option for IBM stocks and their arithmetic average value

Financial derivatives as a tool for protecting volatile portfolios

- A natural question arises:
  Although the time evolution of the asset price S_t as well as its derivative (option) V_t is stochastic (volatile, unpredictable) CAN WE FIND A FUNCTIONAL RELATIONSHIP

$$V_t = V(S_t, t)$$

relating the actual stock price S_t at time t and the price of its derivative (like e.g. an option) V_t?

Financial derivatives as a tool for protecting volatile portfolios

- This was a long standing problem in financial mathematics until 1972. The answer is YES due to pioneering work of M.Scholes, F.Black and R.Merton

Figure: M. S. Scholes a R. C. Merton shortly after they were awarded the Price of the Swedish Bank for Economy in the memory of A. Nobel in 1997.

Financial derivatives as a tool for protecting volatile portfolios

- The Black–Scholes formula

$$V = V(S, t; T, E, r, \sigma)$$

where $S = S_t$ is the spot (actual) price of an asset, $V = V_t$ is a the spot price of the option (call or put) at time $0 \le t \le T$. Here T is the time of maturity, E is the exercise price, $r > 0$ is the interest rate of a secure bond, $\sigma > 0$ is the volatility of underlying stochastic process of the asset price S_t.

- Stochastic differential calculus, Itō's lemma, Itō's integral

- Stochastic process is t - parametric system of random variables $\{X(t), t \in I\}$, where I is an interval or discrete set of indices
- Stochastic process $\{X(t), t \in I\}$ is a Markov process with the property: given a value $X(s)$, the subsequent values $X(t)$ for $t > s$ may depend on $X(s)$ but not on preceding values $X(u)$ for $u < s$.

  If $t \geq s$ then for conditional probabilities we have:

$$P(X(t) < x | X(s)) = P(X(t) < x | X(s), X(u))$$

  for any $u \leq s$.

Stochastic differential calculus, Itō's lemma

- a stochastic process $\{X(t), t \geq 0\}$ is called Brownian motion
  i) all increments $X(t+\Delta) - X(t)$ are normally distributed with mean value $\mu\Delta$ and dispersion (or variance) $\sigma^2\Delta$,
  ii) for any division of times $t_0 = 0 < t_1 < t_2 < t_3 < \ldots < t_n$ the increments $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent random variables
  iii) $X(0) = 0$.
- Brownian motion $\{W(t), t \geq 0\}$ with the mean $\mu = 0$ and dispersion $\sigma^2 = 1$ is called Wiener process

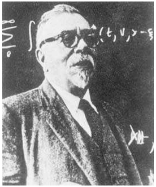


Figure: Norbert Wiener (1884-1964) and Robert Brown (1773-1858).

- Additive (semigroup) property of Brownian motion $\{X(t), t \geq 0\}$

let $0 = t_0 < t_1 < \ldots < t_n = t$ be any division of the interval $[0, t]$. Then

$$X(t) - X(0) = \sum_{i=1}^{n} X_i - X_{i-1},$$

Therefore the mean value E and variance Var of the left and right hand side have to be equal.

$$E(X(t) - X(0)) = \mu(t - 0) = \mu t\,.$$

On the other side we have due to linearity of the mean value operator

$$E\left(\sum_{i=1}^{n} X_i - X_{i-1}\right) = \sum_{i=1}^{n} E(X_i - X_{i-1}) = \sum_{i=1}^{n} \mu(t_i - t_{i-1}) = \mu t$$

- In order to verify the equality we had to require that increments $X_i - X_{i-1}$ have the mean value $E(X_i - X_{i-1}) = \mu(t_i - t_{i-1})$

- Additive (semigroup) property of Brownian motion $\{X(t), t \geq 0\}$

For dispersions of the random variables $X(t) - X(0)$ and $\sum_{i=1}^{n}(X(t_i) - X(t_{i-1}))$ we have

$$Var(X(t) - X(0)) = \sigma^2(t-0) = \sigma^2 t\,.$$

Recall that for random independent variables A, B it holds: $Var(A+B) = Var(A) + Var(B)$. Hence, assuming independence of increments $X_i - X_{i-1}$ for different $i = 1, 2, ..., n$ we have

$$Var\left(\sum_{i=1}^{n} X(t_i) - X(t_{i-1})\right) = \sum_{i=1}^{n} Var(X(t_i) - X(t_{i-1})) = \sum_{i=1}^{n} \sigma^2(t_i - t_{i-1}$$

- In order to verify the equality we had to require that increments $X(t_i) - X(t_{i-1})$ have the dispersion (variance) $V(X(t_i) - X(t_{i-1})) = \sigma^2(t_i - t_{i-1})$

In summary:

- The Brownian motion $\{X(t), t \geq 0\}$ has the following stochastic distribution:

$$X(t) \sim N(\mu t, \sigma^2 t)$$

  where $N(mean, variance)$ stands for normal random variable with given mean and variance

- The Wiener process $\{W(t), t \geq 0\}$ (here $\mu = 0, \sigma^2 = 1$) has the following stochastic distribution:

$$W(t) \sim N(0, t)$$

  Moreover $dW(t) := W(t + dt) - W(t) \sim N(0, dt)$, i.e.

$$dW(t) := W(t + dt) - W(t) = \Phi\sqrt{dt}$$

  where $\Phi \sim N(0, 1)$.

Stochastic differential calculus, Itō's lemma

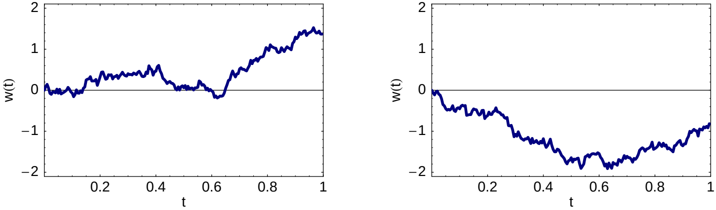


Figure: Two randomly generated samples of a Wiener process.

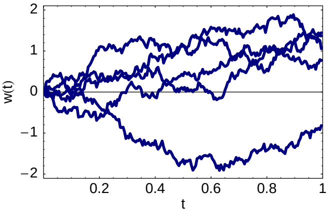


Figure: Five random realizations of a Wiener process alltogether.

Since $W(t) \sim N(0,t)$ we have $Var(W(t)) = t$.

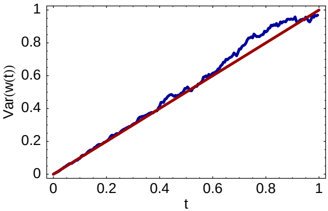


Figure: Time dependence of the variance $Var(W(t))$ of 1000 random realizations of a Wiener process.

Stochastic differential calculus, Itō's lemma

Relation between Brownian and Wiener process:

- For a Brownian motion $\{X(t), t \geq 0\}$ with parameters μ and σ we have by definition $dX(t) = X(t+dt) - X(t) \sim N(\mu dt, \sigma^2 dt)$ Therefore, if we construct the process

$$W(t) = \frac{X(t) - \mu t}{\sigma}$$

we have

$$dW(t) = W(t+dt) - W(t) = \frac{dX(t) - \mu dt}{\sigma} \sim N(0, dt)$$

i.e. $\{W(t), t \geq 0\}$ is a Wiener process

Since $X(t) = \mu t + \sigma W(t)$ we may therefore write a Stochastic differential equation

$$dX(t) = \mu dt + \sigma dW(t),$$

- Geometric Brownian motion

If $\{X(t), t \geq 0\}$ is a Brownian motion with parameters μ and σ we define a new stochastic process $\{Y(t), t \geq 0\}$ by taking

$$Y(t) = y_0 \exp(X(t))$$

where y_0 is a given constant. The process $\{Y(t), t \geq 0\}$ is called Geometric Brownian motion.

- Statistical properties of the Geometric Brownian motion
- let us take for simplicity $y_0 = 1$. Then

$$W(t) = \frac{\ln Y(t) - \mu t}{\sigma}$$

is a Wiener process with $W(t) \sim N(0, t)$, i.e. we know its distribution function

- Statistical properties of the Geometric Brownian motion

Then for the distribution function $G(y,t) = P(Y(t) < y)$ it holds: $G(y,t) = 0$ for $y \leq 0$ (since $Y(t)$ is a positive random variable) and for $y > 0$

$$G(y,t) = P(Y(t) < y) = P\left(W(t) < \frac{-\mu t + \ln y}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\frac{-\mu t + \ln y}{\sigma}} e^{-\xi^2/2t} d\xi$$

- Statistical properties of the Geometric Brownian motion

Since $E(Y(t)) = \int_{-\infty}^{\infty} yg(y,t)\,dy$ and $E(Y(t)^2) = \int_{-\infty}^{\infty} y^2 g(y,t)\,dy$, where $g(y,t) = \frac{\partial}{\partial y} G(y,t)$, we can calculate

$$
\begin{aligned}
E(Y(t)) &= \int_{-\infty}^{\infty} yg(y,t)\,dy = \int_{0}^{\infty} yg(y,t)\,dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^{\infty} y e^{-\frac{(-\mu t + \ln y)^2}{2\sigma^2 t}} \frac{1}{\sigma y}\,dy \\
&\qquad (\xi = (-\mu t + \ln y)/(\sigma\sqrt{t})) \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2} + \sigma\sqrt{t}\xi}\,d\xi = \frac{e^{\mu t + \frac{\sigma^2}{2} t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\xi - \sigma\sqrt{t})^2}{2}}\,d\xi \\
&= e^{\mu t + \frac{\sigma^2}{2} t}.
\end{aligned}
$$

Stochastic differential calculus, Itō's lemma

- Naive (and also wrong) formal calculation

Since $Y(t) = \exp(X(t))$ where $dX(t) = \mu dt + \sigma dW(t)$ we have

$$dY(t) = (\exp(X(t)))' dX(t) = \exp(X(t)) dX(t)$$

and therefore

$$dY(t) = \mu Y(t) dt + \sigma Y(t) dW(t)$$

Hence by taking the mean value operator operator $E(.)$ (it is a linear operator) we obtain

$$dE(Y(t)) = E(dY(t)) = \mu E(Y(t)) dt + \sigma E(Y(t) dW(t)) = \mu E(Y(t)) dt$$

as $Y(t)$ and $dW(t)$ are independent. Solving the differential equation $\frac{d}{dt} E(Y(t)) = \mu E(Y(t))$ yields

$$E(Y(t)) = \exp(\mu t)$$

BUT according to our previous calculus $E(Y(t)) = \exp(\mu t + \frac{\sigma^2}{2} t)$. Where is the mistake?

- The answer is based on Itō's lemma
- We cannot omit stochastic character of the process $\{X(t), t \geq 0\}$ when taking the differential of the COMPOSITE function $\exp(X(t))$!!!

Itō lemma

Let $f(x,t)$ be a C^2 smooth function of x, t variables. Suppose that the process $\{x(t), t \geq 0\}$ satisfies SDE:

$$dx = \mu(x,t)dt + \sigma(x,t)dW,$$

Then the first differential of the process $f = f(x(t), t)$ is given by

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt\,,$$

伊藤 清

Figure: Kiyoshi Itō (1915–2008).

- According to Wikipedia Itō's lemma is the most famous lemma in the world

- Meaning of the stochastic differential equation

$$dx = \mu(x,t)dt + \sigma(x,t)dW,$$

in the sense of Itō.

- Take a time discretization $0 < t_1 < t_2 < \ldots < t_n$. The above SDE is meant in the sense of a limit when the norm $\nu = \max_i |t_{i+1} - t_i| \to 0$ of explicit in time discretization:

$$x(t_{i+1}) - x(t_i) = \mu(x(t_i), t_i)(t_{i+1} - t_i) + \sigma(x(t_i), t_i)(W(t_{i+1}) - W(t_i))$$

- Random variables $x(t_i)$ and $W(t_{i+1}) - W(t_i)$ are independent so does $\sigma(x(t_i), t_i)$ and $W(t_{i+1}) - W(t_i)$. Hence

$$E(\sigma(x(t_i), t_i)(W(t_{i+1}) - W(t_i))) = 0$$

Stochastic differential calculus, Itō's lemma

Intuitive (and not rigorous) proof of Itō's lemma by Taylor series expansion of $f = f(x,t)$ of th 2nd order

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x \partial t}dx\, dt + \frac{\partial^2 f}{\partial t^2}(dt)^2\right) + \text{h.o.t.}$$

Recall that $dw = \Phi\sqrt{dt}$, where $\Phi \approx N(0,1)$,

$$(dx)^2 = \sigma^2(dw)^2 + 2\mu\sigma dw\, dt + \mu^2(dt)^2 \approx \sigma^2 dt + O((dt)^{3/2}) + O((dt)^2)\,.$$

because $E(\Phi^2) = 1$ (dispersion is 1).
Analogously, the term $dx\, dt = O((dt)^{3/2}) + O((dt)^2)$ and thus differential df in the lower order terms dt and dx can be expressed:

$$df = \frac{\partial f}{\partial x}dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right)dt\,.$$

Stochastic differential calculus, Itō's lemma

- Example: Geometric Brownian motion
- $Y(t) = \exp(X(t))$ where $dX(t) = \mu dt + \sigma dW(t)$
  Here $f(x,t) \equiv e^x$ and $Y(t) = f(X(t),t)$. Therefore

$$dY(t) = df = \frac{\partial f}{\partial x} dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt\,.$$

$$= e^{X(t)} dX(t) + \frac{1}{2}\sigma^2 e^{X(t)} dt = (\mu + \frac{1}{2}\sigma^2) Y(t) dt + \sigma Y(t) dW(t)$$

- As a consequence, we have for the mean value $E(Y(t))$

$$dE(Y(t)) = (\mu + \frac{1}{2}\sigma^2) E(Y(t)) dt$$

and so $E(Y(t)) = e^{\mu t + \frac{1}{2}\sigma^2 t}$

- Example: Dispersion of the Geometric Brownian motion
- $Y(t) = \exp(X(t))$ where $dX(t) = \mu dt + \sigma dW(t)$
- Compute $E(Y(t)^2)$. Solution: set $f(x,t) \equiv (e^x)^2 = e^{2x}$.Then

$$dY(t)^2 = df = \frac{\partial f}{\partial x} dx + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt \,.$$

$$= 2e^{2X(t)} dX(t) + \frac{1}{2}\sigma^2 4 e^{2X(t)} dt = 2(\mu+\sigma^2) Y(t)^2 dt + 2\sigma Y(t)^2 dW(t)$$

- As a consequence, we have for the mean value $E(Y(t)^2)$

$$dE(Y(t)^2) = 2(\mu + \sigma^2) E(Y(t)^2) dt$$

and so $E(Y(t)^2) = e^{2\mu t + 2\sigma^2 t}$. Hence

$$D(Y(t)) = E(Y(t)^2) - (E(Y(t))^2 = e^{2\mu t + 2\sigma^2 t}(1 - e^{-\sigma^2 t})$$

Itō's lemma and Fokker–Planck equation

- Suppose that a process $\{x(t), t \geq 0\}$ follows a SDE (It$\bar{0}$'s process)
$$dx = \mu(x,t)dt + \sigma(x,t)dW,$$
where μ a drift function and σ is a volatility of the process.
- Denote by
$$G = G(x,t) = P(x(t) < x \;\mid\; x(0) = x_0)$$
probability distribution function of the process $\{x(t), t \geq 0\}$ starting almost surely from the initial condition x_0
- Then the cummulative distribution function G can be computed from its density function $g = \partial G/\partial x$ where $g(x,t)$ is a solution to the Fokker–Planck equation:
$$\frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right), \quad g(x,0) = \delta(x - x_0).$$

Here $\delta(x-x_0)$ is the Dirac function with support at x_0. It means:

$$\delta(x-x_0) = \begin{cases} 0 & \text{ak} \quad x \neq x_0, \\ +\infty & \text{ak} \quad x = x_0 \end{cases} \quad \text{a} \quad \int_{-\infty}^{\infty} \delta(x-x_0)dx = 1.$$

In our case at the origin $t = 0$ we have

$$G(x,0) = \int_{-\infty}^{x} \delta(\xi - x_0)d\xi = \begin{cases} 0 & \text{ak} \quad x < x_0, \\ 1 & \text{ak} \quad x \geq x_0, \end{cases}$$

so the process $\{x(t), t \geq 0\}$ at $t = 0$ is almost surely equal to x_0.

Intuitive proof of the Fokker-Planck equation:

- Let $V = V(x,t)$ be any smooth function with compact support, i.e. $V \in C_0^\infty(\mathbb{R} \times (0,T))$
- By Itō's lemma we have

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + \mu \frac{\partial V}{\partial x} \right) dt + \sigma \frac{\partial V}{\partial x} dW.$$

- Let E_t be the mean value operator with respect to the random variable having the density function $g(.,t)$, i.e.

$$E_t(V(.,t)) = \int_{\mathbb{R}} V(x,t)\, g(x,t)\, dx$$

Itō's lemma and Fokker–Planck equation

Then

$$dE_t(V(.,t)) = E_t(dV(.,t)) = E_t\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x}\right)dt.$$

because random variables $\sigma(.,t)\frac{\partial V}{\partial x}(.,t)$ and $dW(t)$ are independent and therefore

$$E_t\left(\sigma(.,t)\frac{\partial V}{\partial x}(.,t)dW(t)\right) = 0$$

Itō's lemma and Fokker–Planck equation

- Since $V \in C_0^\infty$ we have $V(x,0) = V(x,T) = 0$ and $V(x,t) = 0$ for $|x| > R$, where $R > 0$ is sufficiently large.
- By integration by parts we obtain

$$
\begin{aligned}
0 &= \int_0^T \frac{d}{dt} E_t(V(.,t))dt = \int_0^T E_t\left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x}\right) dt \\
&= \int_0^T \int_{\mathbb{R}} \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \mu\frac{\partial V}{\partial x}\right) g(x,t)\, dxdt \\
&= \int_0^T \int_{\mathbb{R}} V(x,t)\left(-\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right)\right) dx\, dt.
\end{aligned}
$$

- Since $V \in C_0^\infty(\mathbb{R} \times (0,T))$ is an arbitrary function we obtain the Fokker–Planck equation for the density $g = g(x,t)$:

$$
-\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g\right) - \frac{\partial}{\partial x}\left(\mu g\right) = 0
$$

Itō's lemma and Fokker–Planck equation

- Example: $dx = dW$ and $x(0) = 0$ a.s.
  It means $x(t)$ is a Wiener process
- The Fokker–Planck (diffusion) equation reads as follows:

$$\frac{\partial g}{\partial t} - \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0$$

- Its solution (normalized to be a probabilistic density)

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

is indeed a density function of the random variable $W(t) \sim N(0,t)$

Itō's lemma and Fokker–Planck equation

- Example: $dr = \kappa(\theta - r)dt + \sigma dW$ and and $r(0) = r_0$. This is a so called Ornstein-Uhlenbeck mean reversion process used in the modelling of short rate interest rate stochastic processes
- The Fokker–Planck equation reads as follows:

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial r^2} - \frac{\partial}{\partial r}\left(\kappa(\theta - r)f\right)$$

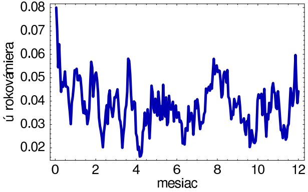
- Its solution (normalized to be a probabilistic density)

$$f(r,t) = \frac{1}{\sqrt{2\pi\bar{\sigma}_t^2}} e^{-\frac{(r-\bar{r}_t)^2}{2\bar{\sigma}_t^2}}$$

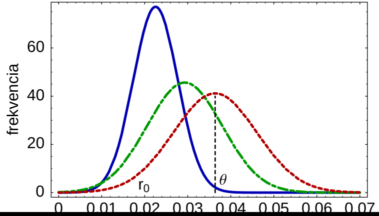
is the density function for the normal random variable $r(t) \sim N(\bar{r}_t, \bar{\sigma}_t^2)$ satisfying the above SDE. Here

$$\bar{r}_t = \theta(1 - e^{-\kappa t}) + r_0 e^{-\kappa t}, \quad \bar{\sigma}_t^2 = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}).$$

- Simulation of the process $r(t)$ satisfying $dr = \kappa(\theta - r)dt + \sigma dW$ and and $r(0) = r_0 = 0.08$. Here $\theta = 0.04$.



- Time steps of the evolution of the density function $f(r,t)$ for various times t. The process $r(t)$ started from $r_0 = 0.02$. The limiting value $\theta = 0.04$



Shift of the density function $f(r,t)$ is due to the drift in the F-P equation

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial r^2} - \frac{\partial}{\partial r}\left(\kappa(\theta - r)f\right)$$

Multidimensional Itō's lemma

- Multidiemnsional stochastic processes

$$dx_i = \mu_i(\vec{x}, t)dt + \sum_{k=1}^{n} \sigma_{ik}(\vec{x}, t)dw_k \,,$$

where $\vec{w} = (w_1, w_2, ..., w_n)^T$ is a vector of Wiener processes having mutually independent increments

$$E(dw_i\, dw_j) = 0 \text{ for } i \neq j\,, \quad E((dw_i)^2) = dt\,.$$

- It can be rewritten in a vector form

$$d\vec{x} = \vec{\mu}(\vec{x}, t)dt + K(\vec{x}, t)d\vec{w}\,,$$

where $\vec{x} = (x_1, x_2, ..., x_n)^T$ and K is an $n \times n$ mattrix

$$K(\vec{x}, t) = (\sigma_{ij}(\vec{x}, t))_{i,j=1,...,n}.$$

Multidimensional Itō's lemma

- Expanding a smooth function $f = f(\vec{x}, t) = f(x_1, x_2, ..., x_n, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ into 2nd order Taylor series yields:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t} dt + \nabla_x f . d\vec{x} \\
&+ \frac{1}{2}\left((d\vec{x})^T \nabla_x^2 f \, d\vec{x} + 2\nabla_x f \frac{\partial f}{\partial t} d\vec{x}\, dt + \frac{\partial^2 f}{\partial t^2} (dt)^2 \right) + \text{ h.o.t.}
\end{aligned}
$$

- The term $(d\vec{x})^T \nabla_x^2 f \, d\vec{x} = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \, dx_j$ can be expanded using the relation between processes x_i and x_j

$$
\begin{aligned}
dx_i \, dx_j &= \sum_{k,l=1}^n \sigma_{ik}\sigma_{jl} dw_k \, dw_l + O((dt)^{3/2}) + O((dt)^2) \\
&\approx (\sum_{k=1}^n \sigma_{ik}\sigma_{jk}) dt + O((dt)^{3/2}) + O((dt)^2)
\end{aligned}
$$

- The multidimensional Itō's lemma gives the SDE for the composite function $f = f(\vec{x}, t)$ in the form:

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2} K : \nabla_x^2 f \; K \right) dt + \nabla_x f d\vec{x}$$

where

$$K : \nabla_x^2 f K = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk}$$

- For the joint density function $g(x_1, x_2, ..., x_n, t)$ for the probability

  $$g(x_1, x_2, ..., x_n, t) = P(x_1(t) = x_1, x_2(t) = x_2, ..., x_n(t) = x_n, t)$$

  conditioned to the initial condition state $x_1(0) = x_1^0, x_2(0) = x_2^0, ..., x_n(0) = x_n^0$ we obtain by following the same procedure of as in the scalar case that:

  $$\frac{\partial g}{\partial t} + \mathrm{div}(\vec{\mu} g) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^{n} \sigma_{ik} \sigma_{jk}\, g \right)$$

  $$g(\vec{x}, 0) = \delta(\vec{x} - \vec{x}^0)$$

  Fokker–Planck equation in the multidimensional case

Multidimensional Itō's lemma and Fokker-Planck equation

- Example: Multidimensional Fokker–Planck equation for a system of uncorrelated SDE's

$$
\begin{aligned}
dx_1 &= \mu_1(\vec{x}, t)dt + \bar{\sigma}_1 dw_1 \\
dx_2 &= \mu_2(\vec{x}, t)dt + \bar{\sigma}_2 dw_2 \\
\vdots & \quad \vdots \qquad\qquad \vdots \\
dx_n &= \mu_n(\vec{x}, t)dt + \bar{\sigma}_n dw_n
\end{aligned}
$$

with mutually independent increments of Wiener processes

$$
E(dw_i \, dw_j) = 0 \text{ for } i \neq j\,, \quad E((dw_i)^2) = dt\,.
$$

- The Fokker–Planck equations reads as follows:

$$
\frac{\partial g}{\partial t} + \mathrm{div}(\vec{\mu} g) = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \left(\bar{\sigma}_i^2 g \right)
$$

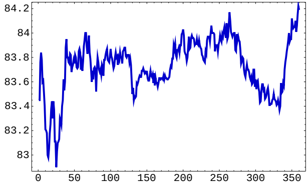
Scalar parabolic reaction–diffusion equation for g

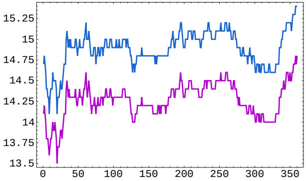
Black–Scholes model for pricing financial derivatives

- Derivation of the Black–Scholes partial differential equation
- the case of Call (or Put) option
- Call option is an agreement (contract) between the writer (issuer) and the holder of an option. It representing the right BUT NOT the obligation to purchase assets at the prescribed exercise price E at the specified time of maturity $t = T$ in the future.
- The question is: What is the price of such an option (option premium) at the time $t = 0$ of contracting. In other words, how much should the holder of the option pay the writer for such a derivative security

Denote

- S - the underlying asset price
- V - the price of a financial derivative (a Call option)
- T - expiration time (time of maturity) of the option contract

S 

V 

Stock prices of IBM (2002/5/2)

Bid and Ask prices of a Call option

Idea

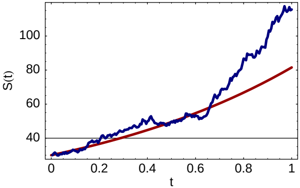
- Look for the price V as a function of s and time $t \in [0, T]$, i.e. $V = V(S, t)$

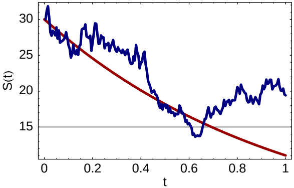
Assumption:

- the underlying asset price follows geometric Brownian motion

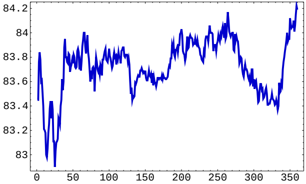
$$dS = \mu S dt + \sigma S dw$$

Simulations of a geometric Brownian motions with $\mu > 0$ (left) and $\mu < 0$ (right)





Real stock prices of IBM (2002/5/2)



Black–Scholes model for pricing financial derivatives

Assumption:

- Fundamental economic balancies
  - conservation of the total portfolio in the book

$$SQ_S + VQ_V + B = 0$$

  - self-financing of the total portfolio in the book

$$SdQ_S + VdQ_V + \delta B = 0$$

where

- Q_S is # of underlying assets with unit price S in the portfolio
- Q_V is # of financial derivatives (options) with unit price V
- B cash money in the portfolio (e.g. bonds, T-bills, etc.)

- dQ_S the change in the number of assets
- dQ_V the change in the number of options
- δB the change in the cash due to buying/selling assets and options