48 (2003) APPLICATIONS OF MATHEMATICS No. 6, 437–453

DIFFUSE-INTERFACE TREATMENT OF THE ANISOTROPIC MEAN-CURVATURE FLOW

Michal Beneš, Praha

Abstract. We investigate the motion by mean curvature in relative geometry by means of the modified Allen-Cahn equation, where the anisotropy is incorporated. We obtain the existence result for the solution as well as a result concerning the asymptotical behaviour with respect to the thickness parameter. By means of a numerical scheme, we can approximate the original law, as shown in several computational examples.

Keywords: mean-curvature flow, phase-field method, FDM, Finsler geometry

MSC 2000: 80A22, 82C26, 35A40

1. Introduction

Mean-curvature flow in relative geometry. The article studies the following motion law for closed hyperplanes in $\mathbb{R}^n$ denoted by Γ:

$$\text{velocity} = -\text{curvature} + \text{forcing} \tag{1}$$

in a certain sense, which is specified below. Both the velocity and the curvature are evaluated with respect to the direction given by a vector locally influenced by the orientation of the Euclidean normal vector to Γ.

One example of the law (1) is represented by the isotropic mean-curvature flow given by the equation

$$v_\Gamma = -\kappa_\Gamma + F, \tag{2}$$

This work was partly supported by the Grant No. 201/01/0676 of the Grant Agency of the Czech Republic. Some computations were performed under the support of the Institute of Computer Science, Academy of Sciences of the Czech Republic, within the project No. A1030103 of the Grant Agency of the Academy of Sciences of the Czech Republic.

in the direction of $\mathbf{n}_\Gamma$ which is the Euclidean normal vector to Γ, while v_Γ is the normal velocity, κ_Γ the mean curvature, and F a forcing term. The equation (2) in the form of the Gibbs-Thompson law is contained in the modified Stefan problem. For details, we refer the reader to [12], [19].

Anisotropic example. One of few cases where the analytical solution is known considers a ball under the relative geometry which shrinks according to (1) with $F = 0$. In this case we have the initial ball with radius r_0, normal velocity $\dot{r}$, actual curvature along the ball of radius r being $\frac{1}{r}$. The equation (1) reads

$$\dot{r} = -\frac{1}{r},$$

and has the solution

$$r(t) = \sqrt{r_0^2 - 2t}. \tag{3}$$

Our aim is to treat the motion law (1) by means of the Allen-Cahn equation, whose solution levelset approximates Γ. The mentioned approach based on non-sharp interpretation of Γ can be traced to [14], [8], [16], [9] or [6]. The physical background is summarized e.g. in [5].

2. Equations

Notation and toolbox. The problem (1) can be analysed in a quite straightforward way, provided we introduce the following framework, based on results published first in [2].

We consider a nonnegative function $\Phi\colon\ \mathbb{R}^n \to \mathbb{R}_0^+$ which is smooth, strictly convex, $\mathcal{C}^2(\mathbb{R}^n \setminus \{0\})$ and satisfying

$$\Phi(t\eta) = |t|\Phi(\eta), \quad t \in \mathbb{R}, \ \ \eta \in \mathbb{R}^n, \tag{4}$$

$$\lambda|\eta| \leqslant \Phi(\eta) \leqslant \Lambda|\eta|, \tag{5}$$

where $\lambda, \Lambda > 0$. The function given by

$$\Phi^0(\eta^*) = \sup\{\eta^* \cdot \eta \mid \Phi(\eta) \leqslant 1\}$$

is its dual. They satisfy the relations

$$\begin{gathered}\Phi^0_\eta(t\eta^*) = \frac{t}{|t|}\Phi^0_\eta(\eta^*), \quad \Phi^0_{\eta\eta}(t\eta^*) = \frac{1}{|t|}\Phi^0_{\eta\eta}(\eta^*), \quad t \in \mathbb{R} - \{0\},\\ \Phi(\eta) = \Phi_\eta(\eta) \cdot \eta, \quad \Phi^0(\eta^*) = \Phi^0_\eta(\eta^*) \cdot \eta^*, \quad \eta, \eta^* \in \mathbb{R}^n,\end{gathered} \tag{6}$$

where the index η means the derivative with respect to η (Φ_η is in fact the total derivative consisting of partial derivatives with respect to components of the vector η). We define the map $T^0\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$
\begin{aligned}
T^0(\eta^*) &:= \Phi^0(\eta^*)\Phi^0_\eta(\eta^*) \quad \text{for } \eta^* \neq 0,\\
T^0(0) &:= 0.
\end{aligned}
$$

It allows to define the Φ-gradient of a smooth function u:

$$
\nabla_\Phi u := T^0(\nabla u) = \Phi^0(\nabla u)\Phi^0_\eta(\nabla u). \tag{7}
$$

If we assume that the hypersurface $\Gamma(t)$ is given by a levelset of the field function $P = P(t,x)$, then

$$
\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^n \mid p(t,\mathbf{x}) = \text{const}\},
$$

and the Φ-normal vector (the Cahn-Hoffmann vector) and the velocity of $\Gamma(t)$ given by a field P are

$$
\mathbf{n}_{\Gamma,\Phi} = -\frac{\nabla_\Phi P}{\Phi^0(\nabla P)} = -\frac{T^0(\nabla P)}{\Phi^0(\nabla P)}, \quad v_{\Gamma,\Phi} = \frac{\partial_t P}{\Phi^0(\nabla P)}.
$$

The anisotropic curvature is given by the formula

$$
\kappa_{\Gamma,\Phi} = \operatorname{div}(\mathbf{n}_{\Gamma,\Phi}).
$$

Example. In 2D, we typically use the dual metric set as

$$
\Phi^0(\eta^*) = \varrho\Psi(\Theta),
$$

where $[\varrho,\Theta]$ are polar coordinates of η^*. Our choice can be $\Psi(\Theta) = 1 + A\sin(m\Theta)$, where $A \geqslant 0$ is the anisotropy strength and $m = 2,3,\ldots$ the order of symmetry. The convexity condition reads $A \leqslant (m^2-1)^{-1}$. In higher dimensions, the anisotropy can explore various norms of the l_p type, as indicated in [2].

Remark. The (strong) monotonicity of the operator T^0 is equivalent to the (strict) convexity of the functional

$$
\int_\Omega \Phi^0(\nabla p)^2 \,\mathrm{d}x.
$$

Using this framework, we can consider the following problem for a nonlinear parabolic equation.

Hamilton-Jacobi equation. Using the above given tools, we can write the law (1) in a more accurate way as

$$v_{\Gamma,\Phi} = -\kappa_{\Gamma,\Phi} + F,$$

in the direction of $\mathbf{n}_{\Gamma,\Phi}$. If the manifold Γ is described as

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^n \mid P(t, \mathbf{x}) = \text{const}\},$$

with a convention

$$\Omega_s(t) = \{\mathbf{x} \in \mathbb{R}^n \mid P(t, \mathbf{x}) > \text{const}\},$$

then we can induce the Hamilton-Jacobi equation

$$\frac{\partial P}{\partial t} = \Phi^0(\nabla P)\nabla \cdot \left(\frac{\nabla_\Phi P}{\Phi^0(\nabla P)}\right) + \Phi^0(\nabla P)F.$$

For the case when $\Phi(\cdot) \equiv |\cdot|$ we obtain the isotropic form of this equation

$$\frac{\partial P}{\partial t} = |\nabla P|\nabla \cdot \left(\frac{\nabla P}{|\nabla P|}\right) + |\nabla P|F,$$

known e.g. from [10], [17].

Wulff shape representing the unit ball under the metric Φ is defined in [13], and can be parametrized as follows:

$$\begin{aligned} \mathcal{W}\colon\ x(\theta) &= \Psi(\theta)\cos\theta - \Psi'(\theta)\sin\theta, \\ y(\theta) &= \Psi(\theta)\sin\theta + \Psi'(\theta)\cos\theta. \end{aligned}$$

In Fig. 1, we show examples of various anisotropies in terms of the Wulff shape. It is worth mentioning that under the law (1), the manifold Γ always tends to approach $\mathcal{W}$ before shrinking or expansion.

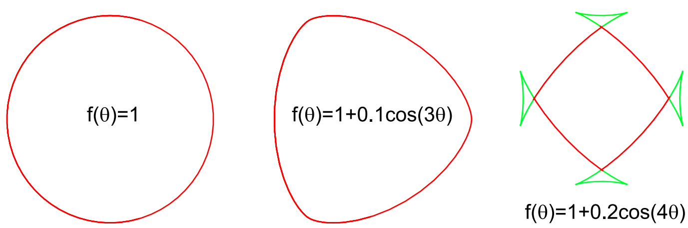

Figure 1. Examples of Wulff shape as the boundary of a convex interior for presented patterns.

IBVP. In analogy with the isotropic motion by mean curvature (e.g., [6]), we propose to use a modified Allen-Cahn equation approximating the manifold Γ through the levelset $\frac{1}{2}$ of its solution. When treating the law (1), we have to incorporate the anisotropy into the equation, which is done in agreement with the results of [2] and [3]. For the sake of simplicity, we restrict ourselves to a two-dimensional rectangular domain and homogeneous Dirichlet boundary conditions.

First, we introduce a rectangular domain $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$, $x = [x_1, x_2] \in \Omega$, and the time variable $t \in (0, T)$. The problem for the unknown function $p = p(t, x)$ reads as follows:

$$
\begin{aligned}
&\xi \frac{\partial p}{\partial t} = \xi \nabla \cdot T^0(\nabla p) + \frac{1}{\xi} f_0(p) + F(u)\xi\Phi^0(\nabla p) && \text{in } (0,T)\times\Omega,\\
&p\big|_{\partial\Omega} = 0 && \text{on } (0,T)\times\partial\Omega,\\
&p|_{t=0} = p_{\text{ini}}(x) && \text{in } \Omega.
\end{aligned} \tag{8}
$$

Here, $\xi > 0$ is a parameter related to the thickness of the interface layer (it is usually set to a value $\ll 1$). The polynomial

$$f_0(p) = ap(1-p)\Big(p - \frac{1}{2}\Big)$$

with $a > 0$ is derived from the double-well potential w_0 as $w_0' = -f_0$. The function $F = F(x)$ is bounded and continuous. The function p_{ini} is the initial condition. We refer the reader to [5], [4] for details concerning the isotropic version of the equation and the physical background.

As usual, we introduce the following notations

$$
\begin{aligned}
(u, v) &= \int_\Omega u(x)v(x)\,\mathrm{d}x,\\
\|u\| &= \sqrt{\int_\Omega u(x)^2\,\mathrm{d}x} \quad \text{for } u, v \in \mathrm{L}_2(\Omega),\\
(\nabla u, \nabla v) &= \int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x,\\
\|\nabla u\| &= \sqrt{\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x} \quad \text{for } u, v \in \mathrm{H}_0^1(\Omega).
\end{aligned}
$$

We also notice that the assumptions on F imply that there is a constant $C_F > 0$ such that $|F| \leqslant C_F$. Our existence and uniqueness result is contained in the following theorem.

Theorem 1. *If* $p_{\mathrm{ini}} \in \mathrm{H}_0^1(\Omega)$ *and* ξ *remains fixed then there is a unique solution* $p^\xi \in \mathrm{L}_2(0,T;\mathrm{H}_0^1(\Omega))$ *of the weak problem*

$$\xi \frac{\mathrm{d}}{\mathrm{d}t}(p^\xi, q) + \xi(T^0(\nabla p^\xi), \nabla q) = \frac{1}{\xi}(f_0(p^\xi), q) + (F\xi\Phi^0(\nabla p^\xi), q), \tag{9}$$
$$p^\xi(0) = p_{\mathrm{ini}},$$

a.e. *in* $(0,T)$, $\forall q \in \mathcal{D}(\Omega)$ *for which*

$$p^\xi \in \mathrm{L}_2(0,T;H^2(\Omega)\cap \mathrm{H}_0^1(\Omega)),$$
$$\frac{\partial p^\xi}{\partial t} \in \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega)).$$

Additionally, the proof of the theorem provides suitable apriori estimates allowing to show

Theorem 2. *Let* p_ξ *be the solution of the weak problem with the initial data satisfying* $E_\xi[p_\xi](0) < M_0$ *independently of* ξ, *and let*

$$\int_\Omega |p_\xi(0,x) - v_0(x)|\,\mathrm{d}x \to 0$$

as $\xi \to 0$, *for a function* $v_0 \in \mathrm{L}_1(\Omega)$. *Then for any sequence* ξ_n *tending to* 0 *there is a subsequence* $\xi_{n'}$ *such that*

$$\lim_{\xi_{n'}\to 0} p_{\xi_{n'}}(t,x) = v(t,x)$$

is defined a.e. in $(0,T)\times\Omega$. *The function* v *reaches values* 0 *and* 1 *and satisfies*

$$\int_\Omega |v(t_1,x) - v(t_2,x)|\,\mathrm{d}x \leqslant C|t_2 - t_1|^{1/2},$$

where $C > 0$ *is a constant, and*

$$\sup_{t\in\langle 0,T\rangle} \int_\Omega |\nabla v|\,\mathrm{d}x \leqslant C_1$$

in the sense of $\mathrm{BV}(\Omega)$, *where* $C_1 > 0$ *is a constant. The initial condition is*

$$\lim_{t\to 0_+} v(t,x) = v_0(x)$$

a.e.

This theorem together with formal asymptotic expansions (see [4] for the isotropic analogue) indicate that the solution p through its levelset $\frac{1}{2}$ approaches the problem

$$v_{\Gamma,\Phi} = -\kappa_{\Gamma,\Phi} + F$$

in the direction of Cahn-Hoffmann vector $\mathbf{n}_{\Gamma,\Phi}$. We sketch this relationship in Fig. 2.

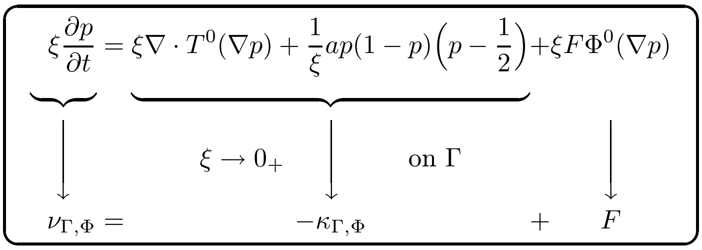

Figure 2. Schematic relationship to the original motion law.

Due to remarks in [2], we keep in mind that the choice $\Phi^0(\eta^*) = \varrho\Psi(\Theta)$ leads to the law

$$v_\Gamma = -\Psi(\Psi + \Psi'')\kappa_\Gamma + \Psi F$$

in the direction of the Euclidean unit normal vector.

3. Proofs of the statements

Proof of Theorem 1. We derive a sequence of approximate solutions to the original problem (9) by means of the Faedo-Galerkin method. Assume that there is an orthonormal basis $\{v_i\}_{i\in\mathbb{N}}$ of the Hilbert space $\mathrm{L}_2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$ coupled with homogeneous Dirichlet boundary conditions.

Additionally, we assume that $(\forall\, i \in \mathbb{N})(v_i \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega}))$. The corresponding eigenvalues are denoted by $\{\lambda_i\}_{i\in\mathbb{N}}$. Let $V_m = \operatorname{span}\{v_i\}_{i\in\mathbb{N}_m}$ be a finite-dimensional subspace ($\mathbb{N}_m = \{1,\ldots,m\}$) and $\mathcal{P}_m\colon \mathrm{L}_2(\Omega) \to V_m$ the L_2-projection operator (coinciding with the H^1-projector). We seek for a solution p^m from $\langle 0,T)$ to V_m of an auxiliary problem

$$(10) \qquad \xi^2 \frac{\mathrm{d}}{\mathrm{d}t}(p^m, v_j) + \xi^2(T^0(\nabla p^m), \nabla v_j) = (f_0(p^m), v_j) + \xi^2(F\Phi^0(\nabla p^m), v_j)$$
$$\text{a.e. in } (0,T), \ \ \forall\, j = 1,\ldots,m,$$
$$p^m(0) = \mathcal{P}_m p_0.$$

We use the basis functions of V_m to express the solution of (10) as

$$p^m(t) = \sum_{i\in\mathbb{N}_m} \gamma_i^m(t) v_i$$

and to obtain a system of ordinary differential equations for the unknown functions of time γ_i^m using (10). We follow the procedure of the compactness method (e.g., see [18]), show that the solution of (10) is defined on $(0,T)$ for $T>0$ and show an appropriate convergence of p^m. For this purpose, we prove an *a priori* estimate by multiplying (10) by $\mathrm{d}\gamma_j^m/\mathrm{d}t$ and summing for $j\in\mathbb{N}_m$:

$$\xi^2\Big\|\frac{\partial p^m}{\partial t}\Big\|^2+\frac{\xi^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^0(\nabla p^m)^2,1)+\frac{\mathrm{d}}{\mathrm{d}t}(w_0(p^m),1)=\xi^2\Big(F\Phi^0(\nabla p^m),\frac{\partial p^m}{\partial t}\Big),$$

where $w_0'=-f_0$, and where the property (6) was important to use:

$$\Phi^0(\nabla p^m)\Phi^0_\eta(\nabla p^m)\cdot\nabla p^m=\Phi^0(\nabla p^m)^2.$$

We use the Schwarz and Young inequalities, estimate $|F|\leqslant C_F$, so that we finally obtain the inequality

$$\frac{1}{2}\xi^2\Big\|\frac{\partial p^m}{\partial t}\Big\|^2+\frac{\xi^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^0(\nabla p^m)^2,1)+\frac{\mathrm{d}}{\mathrm{d}t}(w_0(p^m),1)\leqslant\frac{C_F^2}{2}\xi^2(\Phi^0(\nabla p^m)^2,1).$$

We integrate over $(0,t)$ and subsequently over $(0,T)$, and get

$$\begin{aligned}\Big(\frac{\xi^2}{2}(\Phi^0(\nabla p^m)^2,1)+(w_0(p^m),1)\Big)(t)&\\ \leqslant\Big(\frac{\xi^2}{2}(\Phi^0(\nabla p^m)^2,1)+(w_0(p^m),1)\Big)(0)\exp\Big(\frac{C_F^2}{2}t\Big),&\end{aligned}\tag{11}$$

$$\begin{aligned}\int_0^T\Big(\frac{1}{2}\xi^2\Big\|\frac{\partial p^m}{\partial t}\Big\|^2\Big)(t)\,\mathrm{d}t+\Big(\frac{\xi^2}{2}(\Phi^0(\nabla p^m)^2,1)+(w_0(p^m),1)\Big)(T)&\\ \leqslant\Big(\frac{\xi^2}{2}(\Phi^0(\nabla p^m)^2,1)+(w_0(p^m),1)\Big)(0)&\\ +\frac{C_F^2}{2}\int_0^T\Big(\frac{\xi^2}{2}(\Phi^0(\nabla p^m)^2,1)+(w_0(p^m),1)\Big)(t)\,\mathrm{d}t&\\ \leqslant\Big(\frac{\xi^2}{2}(\Phi^0(\nabla p^m)^2,1)+(w_0(p^m),1)\Big)(0)\exp\Big(\frac{C_F^2}{2}T\Big).&\end{aligned}\tag{12}$$

The assumption of the theorem together with the coincidence of projectors in L_2 and H^1 imply that $\nabla\mathcal{P}_m p_0\in\mathrm{L}_2(\Omega;\mathbb{R}^n)$ and $\mathcal{P}_m p_0$ in $\mathrm{L}_4(\Omega)$ are bounded independently of m.

Consequently, the inequality (5) implies that, independently of m, ∇p^m are bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_2(\Omega))$, and p^m are bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_s(\Omega))$ for each finite time $T>0$ and for any $1\leqslant s\leqslant 4$. The estimate (12) says that $\partial p^m/\partial t$ are bounded in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ for each finite time $T>0$, independently of m.

Therefore, we are able to pass to a weak limit $p^{m'} \rightharpoonup p$ in $\mathrm{L}_2(0,T;\mathrm{H}_0^1(\Omega)\cap\mathrm{L}_4(\Omega))$ via a subsequence m', and thanks to the compact-imbedding theorem with the assumptions

$$\begin{aligned} \{p^m\}_{m=1}^{\infty} &\text{ bounded in } \mathrm{L}_4(0,T;\mathrm{H}_0^1(\Omega)\cap\mathrm{L}_4(\Omega)),\\ \Big\{\frac{\partial p^m}{\partial t}\Big\}_{m=1}^{\infty} &\text{ bounded in } \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega)), \end{aligned}$$

also to the strong limit p in $\mathrm{L}_4(0,T;\mathrm{L}_4(\Omega))$. Such a choice is useful when treating the nonlinear term $f_0(p^m)$, where we apply the Aubin lemma to get weak convergence to $f_0(p)$ in $\mathrm{L}_{\frac43}(0,T;\mathrm{L}_{\frac43}(\Omega))$. We investigate strong convergence of gradients. $\square$

Lemma 1. *The sequence $\nabla p^{m'}$ converges strongly to ∇p in $L_2(0,T;\mathrm{L}_2(\Omega,\mathbb{R}^n))$.*

P r o o f. We multiply the equation (10) by $\gamma_i^m-\gamma_i$ where $p=\sum\limits_{i\in\mathbb{N}}\gamma_i v_i$, sum over $i\in\mathbb{N}$ and integrate over $(0,T)$:

$$\begin{aligned} &\xi^2\int_0^T\Big(\frac{\partial p^{m'}}{\partial t},p^{m'}-p)\,\mathrm{d}t+\xi^2\int_0^T(T^0(\nabla p^{m'}),\nabla(p^{m'}-p))\,\mathrm{d}t\\ &\quad=\int_0^T(f_0(p^{m'}),p^{m'}-p)\,\mathrm{d}t+\xi^2\int_0^T(F\Phi^0(\nabla p^{m'}),p^{m'}-p)\,\mathrm{d}t. \end{aligned} \tag{13}$$

We add and subtract a term

$$\xi^2\int_0^T(T^0(\nabla p),\nabla(p^{m'}-p))\,\mathrm{d}t$$

to the equality (13) knowing that it tends to 0 as

$$\nabla(p^{m'}-p)\to 0$$

weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega,\mathbb{R}^n))$. We also recall that

$$p^{m'}-p\to 0$$

strongly in $\mathrm{L}_4(0,T;\mathrm{L}_4(\Omega))$, if $m'\to\infty$. Then we have

$$\begin{aligned} &\xi^2\int_0^T(T^0(\nabla(p^{m'})-T^0(p)),\nabla(p^{m'}-p))\,\mathrm{d}t\\ &\quad=-\xi^2\int_0^T\Big(\frac{\partial p^{m'}}{\partial t},p^{m'}-p\Big)\,\mathrm{d}t+\int_0^T(f_0(p^{m'}),p^{m'}-p)\,\mathrm{d}t\\ &\qquad+\xi^2\int_0^T(F(u^{m'})\Phi^0(\nabla p^{m'}),p^{m'}-p)\,\mathrm{d}t+\xi^2\int_0^T(T^0(\nabla p),\nabla(p^{m'}-p))\,\mathrm{d}t. \end{aligned}$$

As all terms on the right-hand side tend to 0 if $m' \to \infty$, we see that

$$\int_0^T (T^0(\nabla(p^{m'}) - T^0(p)), \nabla(p^{m'} - p))\,\mathrm{d}t \to 0.$$

The strong monotonicity of T^0,

$$T^0(\nabla(p^{m'}) - T^0(p)), \nabla(p^{m'} - p)) \geqslant c_0 \|\nabla(p^{m'} - p)\|^2,$$

then yields the strong convergence of $\nabla p^{m'}$ to ∇p in $\mathrm{L}_2(0, T; \mathrm{L}_2(\Omega, \mathbb{R}^n))$. □

Since the operator T^0 is (strongly) monotone, we also observe that the term $T^0(\nabla p^{m'})$ converges weakly to $T^0(\nabla p)$ in $\mathrm{L}_2(0, T; \mathrm{L}_2(\Omega; \mathbb{R}^n))$. We are therefore able to pass to the limit in the equation (10) as in [15] to show that u and p is the solution of the problem (9).

We prove **uniqueness** of the solution of (9). We consider two solutions of the problem (9), denoted by p_1 and p_2. Subtracting the corresponding systems of equations and denoting $p_{12} = p_1 - p_2$, multiplying the equation by p_{12}, we have

$$\begin{aligned} &\frac{1}{2}\xi^2 \frac{\mathrm{d}}{\mathrm{d}t}\|p_{12}\|^2 + \xi^2(T^0(\nabla p_1) - T^0(\nabla p_2), \nabla p_{12}) \\ &\qquad = (f_0(p_1) - f_0(p_2), p_{12}) + \xi^2(F\Phi^0(\nabla p_1) - F\Phi^0(\nabla p_2), p_{12}) \text{ in } (0, T), \\ &\quad p_{12}(0) = 0. \end{aligned}$$

We notice that thanks to the shape of f_0 we have

$$(f_0(p_1) - f_0(p_2), p_{12}) \leqslant \|p_{12}\|^2,$$

and the properties of Φ^0 yield

$$|\Phi^0(\nabla p_1) - \Phi^0(\nabla p_2)| \leqslant \Phi^0(\nabla p_{12}).$$

We then have

$$\begin{aligned} &\frac{1}{2}\xi^2 \frac{\mathrm{d}}{\mathrm{d}t}\|p_{12}\|^2 + \xi^2(T^0(\nabla p_1) - T^0(\nabla p_2), \nabla p_{12}) \\ &\qquad = \|p_{12}\|^2 + \xi^2 C_F(\Phi^0(\nabla p_{12}), 1)\|p_{12}\| \text{ in } (0, T), \\ &p_{12}(0) = 0. \end{aligned}$$

For the term with the operator T^0 we use the strong monotonicity with $c_0 > 0$,

$$(T^0(\nabla p_1) - T^0(\nabla p_2), \nabla p_{12}) \geqslant c_0\|\nabla p_{12}\|^2,$$

and the boundedness of Φ^0 by (5) to get the uniqueness result by means of the Gronwall lemma. □

P r o o f of Theorem 2 is completed by using the results of [7] and [6], for which we use the apriori estimate obtained in the previous proof

$$E_\xi[p^\xi](t) \leqslant E_\xi[p^\xi](0)\exp\Big\{\frac{C_F^2}{2}t\Big\} \quad t \in (0,T),$$

where we denoted

$$E_\xi[p^\xi](t) = \int_\Omega \Big[\xi\frac{1}{2}\Phi^0(\nabla p^\xi)^2 + \frac{1}{\xi}w_0(p^\xi)\Big]\,\mathrm{d}x.$$

Additionally, there is an estimate for the time derivative:

$$\frac{1}{2}\xi\int_0^T \|\partial_t p^\xi\|^2\,\mathrm{d}t + E_\xi[p^\xi](T) \leqslant C_T E_\xi[p^\xi](0).$$

□

R e m a r k. We observe that the condition of strong monotonicity is important in the above given proof. Breaking this condition (increasing the anisotropy strength) leads to the investigation of phenomena known as the "crystalline" case. Recently, several new results have been obtained, see [11], [1].

4. Computational results

For the numerical solution of the problem (8), we use the method of lines. First, we introduce the following notation:

$$\mathbf{h} = (h_1,h_2), \quad h_1 = \frac{L_1}{N_1}, \; h_2 = \frac{L_2}{N_2},$$
$$x_{ij} = [x_{ij}^1, x_{ij}^2], \quad u_{ij} = u(x_{ij}),$$
$$\omega_{\mathbf{h}} = \{[ih_1, jh_2] \mid i = 1,\dots,N_1-1;\; j = 1,\dots,N_2-1\},$$
$$\bar\omega_{\mathbf{h}} = \{[ih_1, jh_2] \mid i = 0,\dots,N_1;\; j = 0,\dots,N_2\},$$
$$\gamma_{\mathbf{h}} = \bar\omega_{\mathbf{h}} - \omega_{\mathbf{h}},$$
$$u_{\bar x_1,ij} = \frac{u_{ij}-u_{i-1,j}}{h_1}, \quad u_{x_1,ij} = \frac{u_{i+1,j}-u_{ij}}{h_1},$$
$$u_{\bar x_2,ij} = \frac{u_{ij}-u_{i,j-1}}{h_2}, \quad u_{x_2,ij} = \frac{u_{i,j+1}-u_{ij}}{h_2},$$
$$u_{\bar x_1 x_1,ij} = \frac{1}{h_1^2}(u_{i+1,j} - 2u_{ij} + u_{i-1,j}),$$

and

$$\overline{\nabla}_h u = [u_{\bar x_1}, u_{\bar x_2}], \quad \nabla_h u = [u_{x_1}, u_{x_2}],$$
$$\Delta_h u = u_{\bar x_1 x_1} + u_{\bar x_2 x_2}.$$

Then the semi-discrete scheme has on $\omega_{\mathbf{h}}$ the form

$$\xi^2 \dot{p}^h = \xi^2 \nabla_h \cdot T^0(\overline{\nabla}_h p^h) + f_0(p^h) + \xi^2 \Phi^0(\nabla_h p^h) F, \tag{14}$$
$$p^h|_{\gamma_{\mathbf{h}}} = 0, \quad p^h(0) = \mathcal{P}_h p_{\mathrm{ini}},$$

where its solution is a map $p^h\colon \langle 0, T\rangle \to \mathcal{H}_{\mathbf{h}}$ and $\mathcal{P}_h\colon \mathcal{C}(\overline{\Omega}) \to \mathcal{H}_{\mathbf{h}}$ is a restriction operator.

Convergence results. For the purpose of computational investigation of the convergence towards the sharp-interface description of the flow, we define a problem

$$v_{\Gamma,\Phi} = -\kappa_{\Gamma,\Phi},$$
$$\Gamma|_{t=0} = \mathcal{W},$$

the analytical solution of which is known (see (3)). We compare it with the solution of the problem (9) with the initial condition such that

$$\left\{x \in \Omega | p_{\mathrm{ini}}(x) = \frac{1}{2}\right\} = \mathcal{W}.$$

We measure convergence of the numerically obtained levelsets of (14) towards the analytical ones in terms of the Hausdorff distance. The results are presented in Figs. 3, 4 and in Tab. 1. Here we decrease ξ and the mesh size h simultaneously, observe the number of degrees of freedom, the final time step, and also the CPU consumption (on a PentiumIII 700 MHz Redhat 6.2 Linux computer with the Intel Fortran 6.0 compiler). The experimental order of convergence is the exponent given by the formula

$$\mathrm{Error}_2/\mathrm{Error}_1 = (\mathrm{DoF}_2/\mathrm{DoF}_1)^{\mathrm{EOC}}.$$

Mesh h	regul. ξ	final time step	AC DoF	$\mathrm{L}_\infty(0,T;\mathcal{H})$ error	CPU AC	EOC
0.06	0.2	0.00044643	3596698	0.00563	17.57	–
0.03	0.15	0.00015692	45123804	0.00176	162.01	0.4597
0.02	0.13	0.00008052	195812820	0.00148	612.68	0.2543
0.015	0.10	0.00004640	614053106	0.00139	1513.92	0.0549

Table 1. Table of convergence parameters.

Simulation of anisotropic motion by mean curvature. We present qualitative results obtained for various situations. In the following figures, we show examples of a mean-curvature flow approximated by the Allen-Cahn equation. In Fig. 5, the

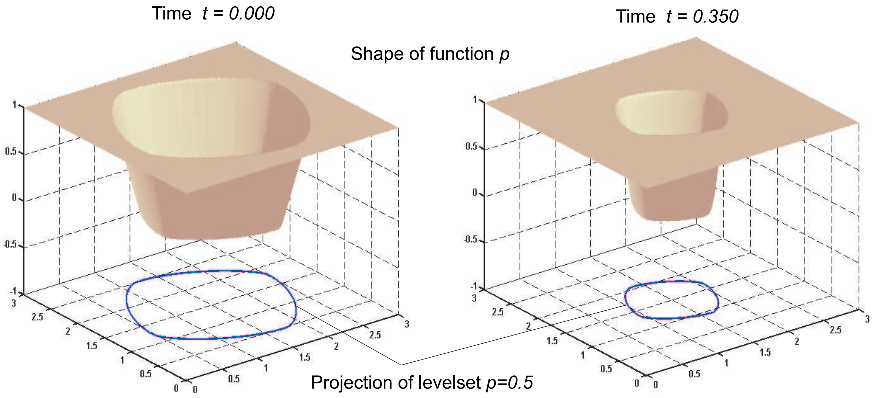

Figure 3. Solution of the Allen-Cahn equation.

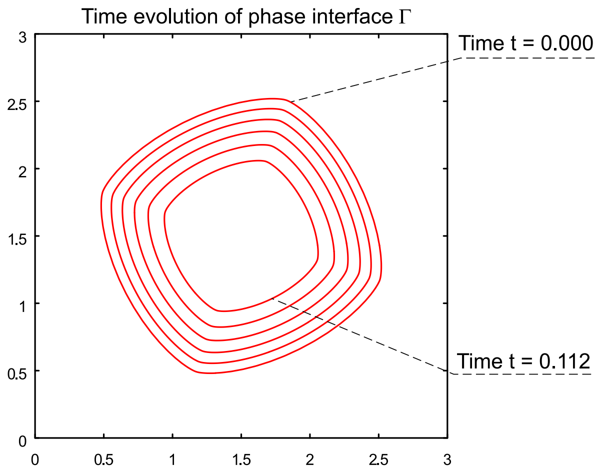

Figure 4. Dynamics of the levelset $\frac{1}{2}$.

initial circular curve is converted to the Wulff shape, and then it is expanded. Fig. 6 shows how the initial nearly rectangular curve is converted again to the Wulff shape which shrinks. Figs. 7 and 8 show a multiple topological change of the initially folded curve thanks to a special choice of $F = F(x)$. Fig. 9 shows the evolution under very strong anisotropy, where the theoretical result cannot be proved in the presented way, but the algorithm still works.

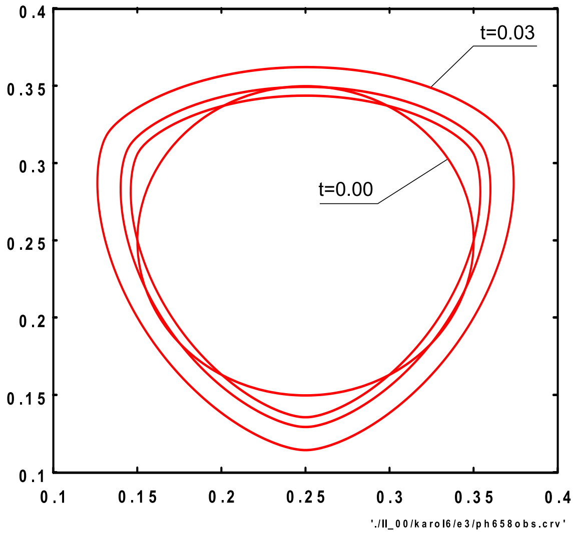

Figure 5. Expansion of a circle at critical radius ($r_0 = 0.1$, $F = 10$) according to $v_\Gamma = -(f(\theta) + f''(\theta))\kappa_\Gamma + F$; $\xi = 0.01$, $h_1 = h_2 = 0.002$, $f(\theta) = 1 + 0.1\cos(3\theta)$.

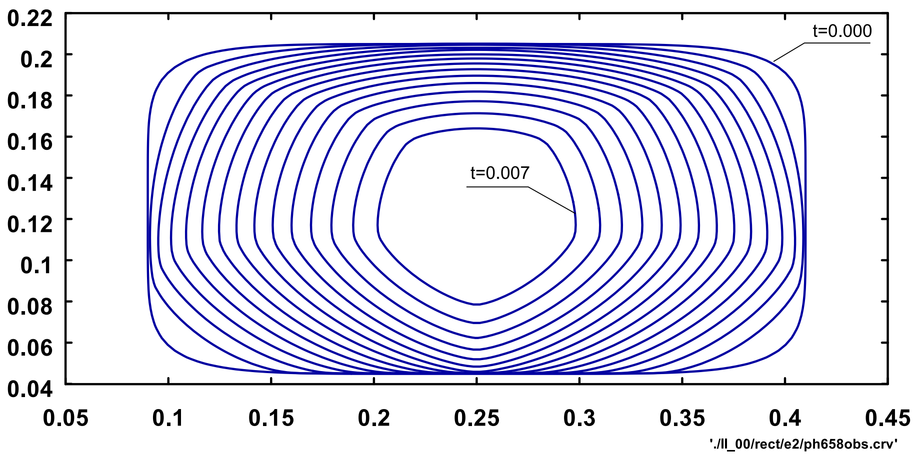

Figure 6. Shrinking of a rounded rectangle according to $v_\Gamma = -(f(\theta) + f''(\theta))\kappa_\Gamma$; $\xi = 0.01$, $h_1 = h_2 = 0.0031$, $f(\theta) = 1 + 0.0375\cos(5\theta)$.

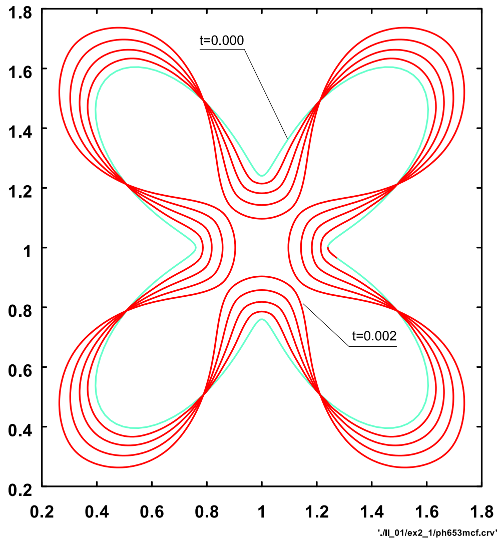

Figure 7. Curve dynamics with space dependent $F = F(x)$ according to $v_\Gamma = -(f(\theta) + f''(\theta))\kappa_\Gamma + F(x)$; $\xi = 0.02$, $h_1 = h_2 = 0.01$, $f(\theta) + f''(\theta) = 1 - 0.8\cos(4\theta)$.

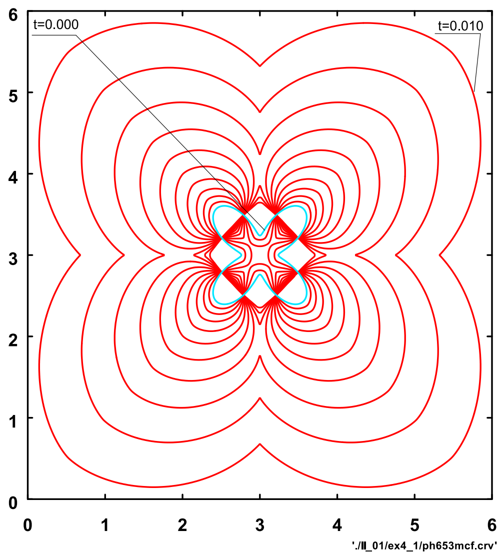

Figure 8. Curve dynamics with space dependent $F = F(x)$ according to $v_\Gamma = -(f(\theta) + f''(\theta))\kappa_\Gamma + F(x)$; $\xi = 0.01$, $h_1 = h_2 = 0.01$, $f(\theta) + f''(\theta) = 1 - 0.8\cos(4\theta)$.

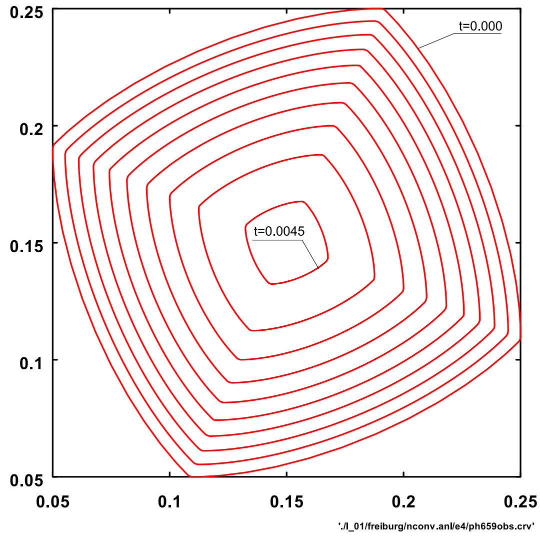

Figure 9. Shrinking of the Wulff shape according to $v_{\Gamma,\Phi} = -\kappa_{\Gamma,\Phi}$ with strong anisotropy; $\xi = 0.01$, $h_1 = h_2 = 0.0012$, $f(\theta) = 1 + 0.1\cos(4(\theta - \frac{\pi}{2}))$.

References

[1] *G. Bellettini, M. Novaga, and M. Paolini*: Characterization of facet-breaking for non-smooth mean curvature flow in the convex case. Interfaces and Free Boundaries *3* (2001), 415–446.

[2] *G. Bellettini, M. Paolini*: Anisotropic motion by mean curvature in the context of Finsler geometry. Hokkaido Math. J. *25* (1996), 537–566.

[3] *M. Beneš*: Anisotropic phase-field model with focused latent-heat release. In: Free Boundary Problems: Theory and Applications II, GAKUTO International Series Mathematical Sciences and Applications, Vol. 14. Chiba, Japan, 2000, pp. 18–30.

[4] *M. Beneš*: Mathematical analysis of phase-field equations with numerically efficient coupling terms. Interfaces and Free Boundaries *3* (2001), 201–221.

[5] *M. Beneš*: Mathematical and computational aspects of solidification of pure substances. Acta Math. Univ. Comenian. *70* (2001), 123–152.

[6] *M. Beneš, K. Mikula*: Simulation of anisotropic motion by mean curvature-comparison of phase-field and sharp-interface approaches. Acta Math. Univ. Comenian. *67* (1998), 17–42.

[7] *L. Bronsard, R. Kohn*: Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. J. Differential Equations *90* (1991), 211–237.

[8] *G. Caginalp*: An analysis of a phase field model of a free boundary. Arch. Rational Mech. Anal. *92* (1986), 205–245.

[9] *C. M. Elliott, M. Paolini, and R. Schätzle*: Interface estimates for the fully anisotropic Allen-Cahn equation and anisotropic mean curvature flow. Math. Models Methods Appl. Sci. *6* (1996), 1103–1118.

[10] *L. C. Evans, J. Spruck*: Motion of level sets by mean curvature I. J. Differential Geom. *33* (1991), 635–681.

[11] *Y. Giga, M. Paolini, and P. Rybka*: On the motion by singular interfacial energy. Japan J. Indust. Appl. Math. *18* (2001), 47–64.
[12] *M. Gurtin*: On the two-phase Stefan problem with interfacial energy and entropy. Arch. Rational Mech. Anal. *96* (1986), 200–240.
[13] *M. Gurtin*: Thermomechanics of Evolving Phase Boundaries in the Plane. Clarendon Press, Oxford, 1993.
[14] *D. A. Kessler, J. Koplik, and H. Levine*: Geometrical models of interface evolution. III. Theory of dendritic growth. Phys. Rev. A *31* (1985), 1712–1717.
[15] *J.-L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod Gauthier-Villars, Paris, 1969.
[16] *T. Ohta, M. Mimura, and R. Kobayashi*: Higher-dimensional localized patterns in excitable media. Physica D *34* (1989), 115–144.
[17] *J. A. Sethian*: Level Set Methods. Cambridge University Press, New York, 1996.
[18] *R. Temam*: Navier-Stokes Equations, Theory and Numerical Analysis. North-Holland, Amsterdam, 1979.
[19] *A. Visintin*: Models of Phase Transitions. Birkhäuser, Boston, 1996.

Author's address: *Michal Beneš*, Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague 2, Czech Republic, e-mail: benes@km1.fjfi.cvut.cz.