Journal of Computational and Applied Mathematics 136 (2001) 163–176

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Nonlinear Galerkin method for reaction–diffusion systems admitting invariant regions

Jan Šembera[a,*], Michal Beneš[b]

[a]Department of Modelling of Processes, Faculty of Mechatronics and Interdisciplinary Engineering Studies, Technical University in Liberec, Hálkova 6, 461 17 Liberec, Czech Republic

[b]Department of Mathematics, Faculty of Nuclear Science and Physical Engineering, Czech Technical University of Prague, Trojanova 13, 120 00 Prague, Czech Republic

Received 25 January 1999; received in revised form 10 May 1999

Abstract

The article presents an analysis of the nonlinear Galerkin method applied to a system of reaction–diffusion equations. If the system admits a bounded invariant region, it is possible to demonstrate the convergence of the approximate solutions to the weak solution of the system. The proof is based on the compactness technique. It is performed for arbitrary ratio of dimensions of the approximation space and of the correction space used in the nonlinear Galerkin method. This fact, generalizing the previously published results, is important for the practical use of the method and allows optimization of the CPU-time consumption of the algorithm. The method is applied to the well-known Brusselator system for which we present an overview of the computational results and our experience with the numerical method used. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Reaction–diffusion equation; Nonlinear Galerkin method; Invariant region; Compactness technique

1. Introduction

Consider a system of differential equations in the form

$$\frac{\partial u}{\partial t} = \boldsymbol{D}\,\Delta u + \boldsymbol{F}(u), \tag{1}$$

where $\boldsymbol{D} \in \mathbb{R}^{d,d}$ denotes a positively definite diagonal matrix, $\boldsymbol{F}$: $\mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz-continuous map (i.e., any restriction to a bounded domain in $\mathbb{R}^d$ is Lipschitz-continuous), $u(t,z)$ is a d-dimensional function of time t ($t \geqslant 0$) and of space z ($z \in \Omega \subset \mathbb{R}^n$).

* Corresponding author.

E-mail address: jan.sembera@vslib.cz (J. Šembera).

PII: S0377-0427(00)00582-3

We solve Eq. (1) in a bounded space domain Ω, having piecewise smooth boundary, and we consider the homogeneous Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \tag{2}$$

and the initial condition

$$u|_{t=0} = u_0. \tag{3}$$

We introduce the space $\boldsymbol{H} := L^2(\Omega; \mathbb{R}^d)$ as a Hilbert space with the scalar product

$$(u, v) \equiv (u, v)_{\boldsymbol{H}} = \sum_{i=1}^{d} (u_i, v_i)_{L^2(\Omega)} = \sum_{i=1}^{d} \int_\Omega u_i v_i,$$

and the space $\boldsymbol{V} := H_0^{(1)}(\Omega; \mathbb{R}^d)$ as a Hilbert space with the scalar product

$$(u, v)_{\boldsymbol{V}} = \sum_{i=1}^{d} (u_i, v_i)_{H_0^{(1)}(\Omega)} = \sum_{i=1}^{d} \int_\Omega \nabla u_i \cdot \nabla v_i,$$

where $u = (u_1, \ldots, u_d)^{\mathrm{T}}$, $v = (v_1, \ldots, v_d)^{\mathrm{T}}$.

Definition 1. Let $u_0 \in \boldsymbol{H}$; then the *weak solution* of problem (1)–(3) on a time interval $(0, T)$ is a mapping $u : (0, T) \to \boldsymbol{V}$ such that it satisfies the following conditions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, w) + (\boldsymbol{D}u, w)_{\boldsymbol{V}} = (\boldsymbol{F}(u), w) \quad \text{a.e. in } (0, T) \quad \forall w \in \mathscr{D}(\Omega),$$

$$u|_{t=0} = u_0. \tag{4}$$

In addition, we assume that problem (1)–(3) has a bounded closed convex *invariant region* $\mathscr{O} \subset \mathbb{R}^d$, which means (see [7, Section 4] or [9]) that if for almost every $z \in \Omega$, the initial condition $u_0(z) \in \mathscr{O}$, then $(\forall z \in \Omega)(u(t, z) \in \mathscr{O})$ for every $t > 0$, for which the solution u exists.

Denote $\boldsymbol{H}(\mathscr{O})$ the space of functions from $\boldsymbol{H}$ for which $u_0(z) \in \mathscr{O}$ for almost every $z \in \Omega$.

2. Nonlinear Galerkin method

The nonlinear Galerkin method described in [4], [5] or [8] is applied to the system (1). We consider an orthonormal basis of the space $\boldsymbol{H}$ composed of eigenvectors of the operator $-\boldsymbol{D}\Delta$ in Ω satisfying the homogeneous boundary condition:

$$\left\{ w_j^{(1)} = \begin{pmatrix} \Phi_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, w_j^{(d)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi_j \end{pmatrix} \right\}_{j=1}^{\infty},$$

where $\Phi_j(z) \in C_0^2(\Omega)$ have the following properties:

$$\forall j \geqslant 1 : -\Delta \Phi_j = \lambda_j \Phi_j,$$

$$i < j \Rightarrow 0 < \lambda_i \leqslant \lambda_j, \quad \lambda_j \stackrel{j\to+\infty}{\longrightarrow} +\infty,$$

$$\forall i, j \geqslant 1 : (\Phi_i, \Phi_j)_{L^2(\Omega)} = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol.

The approximate solution $u_m(t)$ of system (1) in each time t is searched in a dm-dimensional subspace $P_{dm}\boldsymbol{H} = [w_1^{(1)}, \dots, w_1^{(d)}, \dots, w_m^{(1)}, \dots, w_m^{(d)}]_\lambda$ generated by the first dm naturally selected functions of the basis. In addition, we consider a correction term $z_m(t)$ from the space $(P_{dM} - P_{dm})\boldsymbol{H} = [w_{m+1}^{(1)}, \dots, w_{m+1}^{(d)}, \dots, w_M^{(1)}, \dots, w_M^{(d)}]_\lambda$ generated by the next $d(M-m)$ functions of the basis. Let us denote

$$u_m(t) = \sum_{i=1}^{m} \sum_{l=1}^{d} \alpha_i^{(l)}(t) w_i^{(l)}, \quad z_m(t) = \sum_{i=m+1}^{M} \sum_{l=1}^{d} \alpha_i^{(l)}(t) w_i^{(l)}.$$

The equations of the nonlinear Galerkin method are

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_m(t), w_k^{(l)}) + (\boldsymbol{D}u_m, w_k^{(l)})_V = (\boldsymbol{F}(u_m(t) + z_m(t)), w_k^{(l)}), \quad k = 1, \dots, m; \;\; l = 1, \dots, d, \tag{5}$$

$$(\boldsymbol{D}z_m, w_k^{(l)})_V - (\nabla \boldsymbol{F}(u_m(t)) z_m(t), w_k^{(l)}) = (\boldsymbol{F}(u_m(t)), w_k^{(l)}), \quad k = m+1, \dots, M; \;\; l = 1, \dots, d, \tag{6}$$

where $\nabla \boldsymbol{F}$ is the Fréchet derivative of $\boldsymbol{F}$.

The initial condition is given by a projection of u_0:

$$u_m(0) = u_{0m} := P_{dm} u_0. \tag{7}$$

3. Convergence of the nonlinear Galerkin method

According to [1], we define an operator $\boldsymbol{A}$ and a mapping $\tilde{\boldsymbol{G}}$ by the following relations:

$$\boldsymbol{A}u := -\boldsymbol{D}\Delta u + u, \qquad \tilde{\boldsymbol{G}}(u) := u + \boldsymbol{F}(u).$$

The Fréchet derivative of the mapping $\tilde{\boldsymbol{G}}(u)$ can be written as

$$\nabla \tilde{\boldsymbol{G}}(u) = \boldsymbol{Id} + \nabla \boldsymbol{F}(u). \tag{8}$$

In this notation, Eqs. (5) and (6) have the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_m + \boldsymbol{A}u_m = P_{dm} \tilde{\boldsymbol{G}}(u_m + z_m), \tag{9}$$

$$-\boldsymbol{A}z_m + (P_{dM} - P_{dm}) \tilde{\boldsymbol{G}}(u_m) = -(P_{dM} - P_{dm}) \nabla \tilde{\boldsymbol{G}}(u_m) z_m, \tag{10}$$

and Eq. (1) is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}t} u + \boldsymbol{A}u = \tilde{\boldsymbol{G}}(u). \tag{11}$$

The investigation of convergence requires a computation of eigenvalues and eigenvectors of the operator $\boldsymbol{A}$. Obviously, the eigenvectors of $\boldsymbol{A}$ are identical with those of $-\boldsymbol{D}\Delta$, i.e. $\{w_j^{(1)},\dots,w_j^{(d)}\}_{j\geqslant 1}$, and the corresponding eigenvalues are

$$\Lambda_j^{(1)} = D_1\lambda_j + 1,\dots,\Lambda_j^{(d)} = D_d\lambda_j + 1, \quad j \in N,$$

where $\boldsymbol{D} = \operatorname{diag}(D_1,\dots,D_d)$. For further modifications, we introduce the following notation:

$$D_{\min}:=\min\{D_i \mid 1\leqslant i\leqslant d\}, \quad \Lambda_j^{\min}:=\min\{\Lambda_j^{(i)} \mid 1\leqslant i\leqslant d\}.$$

$\boldsymbol{A}$ is a positive self-adjoint operator in $\boldsymbol{H}$. Therefore, for all $p,q \in \operatorname{Def}(\boldsymbol{A}) = \operatorname{Def}(\boldsymbol{A}^*) \subset \operatorname{Def}(\sqrt{\boldsymbol{A}})$, the relation $(\boldsymbol{A}p,q) = (p,\boldsymbol{A}^*q) = (p,\boldsymbol{A}q) = (\sqrt{\boldsymbol{A}}p,\sqrt{\boldsymbol{A}}q)$ holds. In addition, for $q \in \operatorname{Def}(\boldsymbol{A})$:

$$\begin{aligned}||\boldsymbol{A}q||^2 &= ||-\boldsymbol{D}\Delta q||^2 + ||q||^2 - 2(\boldsymbol{D}\Delta q,q)\\ &= ||\boldsymbol{D}\Delta q||^2 + ||q||^2 + 2||\sqrt{\boldsymbol{D}}\nabla q||^2, \end{aligned} \tag{12}$$

$$||\sqrt{\boldsymbol{A}}q||^2 = (\boldsymbol{A}q,q) = (-\boldsymbol{D}\Delta q,q) + ||q||^2 = ||\sqrt{\boldsymbol{D}}\nabla q||^2 + ||q||^2, \tag{13}$$

$$||\sqrt{\boldsymbol{D}}q||^2 \geqslant D_{\min}||q||^2, \tag{14}$$

$$||\boldsymbol{D}q||^2 \geqslant D_{\min}^2||q||^2. \tag{15}$$

Theorem 2. *If Eq.* (1) *admits a closed convex bounded invariant region* $\mathcal{O} \subset \mathbb{R}^d$ *and if the initial condition* u_0 *belongs to* $\boldsymbol{H}(\mathcal{O})\cap\operatorname{Def}(\sqrt{\boldsymbol{A}})$, *then the sequence* $\{u_m\}_{m=1}^{\infty}$ *of the solutions of system* (9) *and* (10) *given by the nonlinear Galerkin method converges strongly to the unique weak solution* $u \in L^2(0,T;\boldsymbol{V})$ *of problem* (1)–(3) *in* $L^2(0,T;\boldsymbol{H})$ *for each* $T>0$, *if* $m \to +\infty$ *and* $M > m$. *Moreover, there exists a sub-sequence* $\{u_{m'}\}_{m'=1}^{\infty}$ *of* $\{u_m\}_{m=1}^{\infty}$ *converging weak-star to the solution* u *in* $L^2(0,+\infty;\boldsymbol{H})$.

Proof. We use the technique of [7]. Let us define the functions on $\mathbb{R}^d$:

$$\phi:=\begin{cases} e^{1/||s||_{\mathbb{R}^d}^2-1} & \text{for } ||s||_{\mathbb{R}^d} < 1,\\ 0 & \text{for } ||s||_{\mathbb{R}^d} \geqslant 1,\end{cases} \qquad \phi_0:=\left(\int_{\mathbb{R}^d}\phi(s)\,\mathrm{d}s\right)^{-1}\phi \in C_0^\infty(\mathbb{R}^d).$$

Let $\mathcal{O}_\varepsilon$ be an open ε-neighbourhood of the region $\mathcal{O}$, i.e. $\mathcal{O}_\varepsilon = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,\mathcal{O}) < \varepsilon\}$. We define, for any ε and set $\mathcal{O}_\varepsilon$, a function ζ as a "mollified" characteristic function of $\mathcal{O}_\varepsilon$: $\zeta(x):=\varepsilon^{-2}\int_{\mathcal{O}_\varepsilon}\phi_0((x-y)/\varepsilon)\,\mathrm{d}y$. Consequently, $\operatorname{supp}\zeta = \mathcal{O}_\varepsilon$ and $(\forall s \in \mathcal{O})(\zeta(s)=1)$ and $(\forall s \in \mathcal{O}_\varepsilon)(\zeta(s)\leqslant 1)$.

We define a mapping $\boldsymbol{G}$ in $\mathbb{R}^d$ as $(\forall q \in \mathbb{R}^d)(\boldsymbol{G}(q):=\zeta(q)\tilde{\boldsymbol{G}}(q))$. Then there exist constants $k_0>0$, and $k_1>0$ such that $(\forall q \in \mathbb{R}^d)(||\boldsymbol{G}(q)||_{\mathbb{R}^d}\leqslant k_0)$, $(\forall q,p \in \mathbb{R}^d)(||\nabla\boldsymbol{G}(q)p||_{\mathbb{R}^d}\leqslant k_1||p||_{\mathbb{R}^d})$ and $(\forall q \in \mathcal{O})(\boldsymbol{G}(q)=\tilde{\boldsymbol{G}}(q))$.

Moreover, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u + \boldsymbol{A}u = \boldsymbol{G}(u). \tag{16}$$

admits the same invariant region as (11), and if, considering the initial condition u_0 with all its values in $\mathcal{O}$, the solution of (11) stays in the invariant region $\mathcal{O}$, where Eq. (16) is identical with Eq. (11), then their solutions will have to coincide.

We prove the convergence of the solutions of the modified equation (16) obtained by the nonlinear Galerkin method, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}u_m + \boldsymbol{A}u_m = P_{dm}\boldsymbol{G}(u_m + z_m), \tag{17}$$

$$\boldsymbol{A}z_m - (P_{dM} - P_{dm})\boldsymbol{G}(u_m) = (P_{dM} - P_{dm})\nabla\boldsymbol{G}(u_m)z_m, \tag{18}$$

to the solution of problem (16). It will imply the convergence of the nonlinear Galerkin method (9) and (10) to the original equation (11), if the initial condition has values in the invariant region $\mathcal{O}$.

3.1. Sequence $\{z_m\}_{m=1}^{+\infty}$

Applying (18) to $\boldsymbol{A}z_m$ and using twice the Young inequality

$$|(f,g)| \leqslant \frac{\varepsilon}{2}||f||^2 + \frac{1}{2\varepsilon}||g||^2, \quad f,g \in \boldsymbol{H}, \tag{19}$$

for $\varepsilon = 2$, we obtain

$$||\boldsymbol{A}z_m||^2 \leqslant ||\boldsymbol{G}(u_m)||^2 + \tfrac{1}{4}||\boldsymbol{A}z_m||^2 + ||\nabla\boldsymbol{G}(u_m)z_m||^2 + \tfrac{1}{4}||\boldsymbol{A}z_m||^2. \tag{20}$$

By relation [1, (1.14)], we observe that

$$||\boldsymbol{A}z_m||^2 \geqslant \Lambda_{m+1}^{\min}||\sqrt{\boldsymbol{A}}z_m||^2 = (1 + D_{\min}\lambda_{m+1})\,||\sqrt{\boldsymbol{A}}z_m||^2. \tag{21}$$

Using (21), Eq. (20) becomes

$$\Lambda_{m+1}^{\min}||\sqrt{\boldsymbol{A}}z_m||^2 \leqslant 2(||\boldsymbol{G}(u_m)||^2 + ||\nabla\boldsymbol{G}(u_m)z_m||^2).$$

The boundedness of $\boldsymbol{G}$ and of $\nabla\boldsymbol{G}$ implies that

$$||\sqrt{\boldsymbol{A}}z_m||^2 \leqslant \frac{2}{1 + D_{\min}\lambda_{m+1}}(k_0^2 + k_1^2||z_m||^2). \tag{22}$$

The left-hand side of this inequality can be modified using (13), (14), and the Poincaré inequality with the constant C_Ω to obtain

$$\left(\frac{D_{\min}}{C_\Omega} + 1\right)||z_m||^2 \leqslant \frac{2}{1 + D_{\min}\lambda_{m+1}}(k_0^2 + k_1^2||z_m||^2).$$

Finally, we obtain

$$||z_m||^2 \leqslant \frac{2k_0^2}{(D_{\min}/C_\Omega + 1)(1 + D_{\min}\lambda_{m+1}) - 2k_1^2}.$$

Consequently, the fact that $\lambda_j \stackrel{j\to+\infty}{\longrightarrow} +\infty$ implies

$$||z_m||^2 \stackrel{m\to+\infty}{\longrightarrow} 0,$$

$$z_m(t) \stackrel{m\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H} \text{ uniformly with respect to } t \geqslant 0.$$

3.2. Sequence $\{u_m\}_{m=1}^{+\infty}$

Assume that $u_0 \in \mathrm{Def}(\sqrt{\boldsymbol{A}})$. Then, the following relations are a direct consequence of the Bessel inequality:

$$||u_m(0)|| = ||u_{0m}|| = ||P_{dm}u_0|| \leqslant ||u_0||,$$

$$||\sqrt{\boldsymbol{A}}u_m(0)|| = ||\sqrt{\boldsymbol{A}}u_{0m}|| = ||\sqrt{\boldsymbol{A}}P_{dm}u_0|| \leqslant ||\sqrt{\boldsymbol{A}}u_0||.$$

We apply (17) on $\boldsymbol{A}u_m$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{\boldsymbol{A}}u_m, \sqrt{\boldsymbol{A}}u_m) = (-\boldsymbol{A}u_m, \boldsymbol{A}u_m) + (\boldsymbol{G}(u_m + z_m), \boldsymbol{A}u_m).$$

Using Young inequality (19) for $\varepsilon = 1$, the boundedness of $\boldsymbol{G}$, and the properties of the operators $\boldsymbol{A}$ and $\sqrt{\boldsymbol{A}}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\sqrt{\boldsymbol{A}}u_m||^2 + ||\boldsymbol{A}u_m||^2 \leqslant \tfrac{1}{2}k_0^2 + \tfrac{1}{2}||\boldsymbol{A}u_m||^2,$$

and after a simple modification:

$$\frac{\mathrm{d}}{\mathrm{d}t}||\sqrt{\boldsymbol{A}}u_m||^2 + ||\boldsymbol{A}u_m||^2 \leqslant k_0^2. \tag{23}$$

Integrating (23) over $t \in (0, T)$ and neglecting the positive norm on the left-hand side, we obtain

$$\int_0^T ||\boldsymbol{A}u_m(t)||^2\,\mathrm{d}t \leqslant k_0^2 T + ||\sqrt{\boldsymbol{A}}u_m(0)||^2 \leqslant k_0^2 T + ||\sqrt{\boldsymbol{A}}u_0||^2$$

$$\Rightarrow (\forall T > 0)(\{\boldsymbol{A}u_m\}_m \text{ bounded in } L^2(0, T; \boldsymbol{H})).$$

Following Eq. (17), the boundedness of $\{\boldsymbol{A}u_m\}_m$ and $\{\boldsymbol{G}(u_m + z_m)\}_m$, we observe that

$$(\forall T > 0)\left(\left\{\frac{\mathrm{d}}{\mathrm{d}t}u_m\right\}_m \text{ bounded in } L^2(0, T; \boldsymbol{H})\right). \tag{24}$$

Relations (12) and (13) imply that for $q \in \mathrm{Def}(\boldsymbol{A})$: $||\boldsymbol{A}q||^2 \geqslant ||\sqrt{\boldsymbol{A}}q||^2$. We use this relation to modify inequality (23):

$$\frac{\mathrm{d}}{\mathrm{d}t}||\sqrt{\boldsymbol{A}}u_m||^2 + ||\sqrt{\boldsymbol{A}}u_m||^2 \leqslant k_0^2. \tag{25}$$

The uniform Gronwall lemma implies

$$||\sqrt{\boldsymbol{A}}u_m(t)||^2 \leqslant \mathrm{e}^{-t}[||\sqrt{\boldsymbol{A}}u_m(0)||^2 + k_0^2(\mathrm{e}^t - 1)] \leqslant \mathrm{e}^{-t}[||\sqrt{\boldsymbol{A}}u_0||^2 + k_0^2(\mathrm{e}^t - 1)]$$

$$\Rightarrow \{\sqrt{\boldsymbol{A}}u_m\}_m \text{ bounded in } L^\infty(0, +\infty; \boldsymbol{H}) \subset L^\infty(0, T; \boldsymbol{H}) \subset L^2(0, T; \boldsymbol{H}) \tag{26}$$

$$\Rightarrow (\forall T > 0)(\{u_m\}_m \text{ bounded in } L^2(0, T; \boldsymbol{V})). \tag{27}$$

Consequently, there exists a sub-sequence $\{u_{m'}\}_{m'}$ weakly converging in the space $L^2(0, T; \boldsymbol{V})$. From [6, Theorem 5.1], it follows that for each finite $T > 0$, the space $W_T := \{q \in L^2(0, T; \boldsymbol{V}) | (\mathrm{d}/\mathrm{d}t)q \in L^2(0, T; \boldsymbol{H})\}$ with the norm $||q||_W := ||q||_{L^2(0,T;\boldsymbol{V})} + ||(\mathrm{d}/\mathrm{d}t)q||_{L^2(0,T;\boldsymbol{H})}$ is a Banach space, and it is compactly embedded into $L^2(0, T; \boldsymbol{H})$. Then, following (24) and (27), $\{u_{m'}\}_{m'}$ converges strongly in $L^2(0, T; \boldsymbol{H})$. Let us denote its limit as u.

3.3. Sequence $\{\boldsymbol{G}(u_m + z_m)\}_{m=1}^{+\infty}$

Since $\boldsymbol{G}$ is Lipschitz-continuous, the following relation holds:

$$||\boldsymbol{G}(u_{m'} + z_{m'}) - \boldsymbol{G}(u)||^2_{L^2(0,T;\boldsymbol{H})} = \int_0^T ||\boldsymbol{G}((u_{m'} + z_{m'})(t)) - \boldsymbol{G}(u(t))||^2 \,\mathrm{d}t$$

$$\leqslant \int_0^T \mathscr{L}^2 ||u_{m'}(t) + z_{m'}(t) - u(t)||^2 \,\mathrm{d}t = \mathscr{L}^2 ||u_{m'} + z_{m'} - u||^2_{L^2(0,T;\boldsymbol{H})}$$

$$\leqslant \mathscr{L}^2 \left(||u_{m'} - u||_{L^2(0,T;\boldsymbol{H})} + ||z_{m'}||_{L^2(0,T;\boldsymbol{H})}\right)^2 \overset{m' \to +\infty}{\longrightarrow} 0,$$

where $\mathscr{L}$ is the Lipschitz constant of $\boldsymbol{G}$. Consequently, $\{\boldsymbol{G}(u_{m'} + z_{m'})\}_{m'}$ also converges strongly to $\boldsymbol{G}(u)$ in $L^2(0,T;\boldsymbol{H})$.

3.4. Passage to the limit

For the subscript m', Eq. (17) can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_{m'}, w_j) = (-\boldsymbol{A}u_{m'}, w_j) + (\boldsymbol{G}(u_{m'} + z_{m'}), w_j), \quad j = 1, \ldots, m'.$$

For a fixed positive-finite time T, we multiply the previous relation by a function $\psi \in C^1(0,T)$, for which $\psi(T) = 0$, and integrate by parts over $t \in (0,T)$ to obtain

$$-\psi(0)(u_{m'}(0), w_j) - \int_0^T (u_{m'}(t), w_j) \frac{\mathrm{d}}{\mathrm{d}t}\psi(t)\,\mathrm{d}t$$

$$= \int_0^T [(\boldsymbol{G}(u_{m'}(t) + z_{m'}(t)), w_j) - (\sqrt{\boldsymbol{A}}u_{m'}(t), \sqrt{\boldsymbol{A}}w_j)]\psi(t)\,\mathrm{d}t. \tag{28}$$

Before proceeding with the proof, we summarize:

- By (7), the initial conditions $u_{0m'}$ converge strongly to the initial condition u_0 in $\boldsymbol{H}$.
- The conclusion of Section 3.2 says that $u_{m'}$ converges strongly in the space $L^2(0,T;\boldsymbol{H})$ to the function u.
- By Section 3.3, there is the strong convergence of $\boldsymbol{G}(u_{m'} + z_{m'})$ in the same space to $\boldsymbol{G}(u)$.
- The weak convergence of $u_{m'}$ to u in $L^2(0,T;\boldsymbol{V})$ implies that

$$\int_0^T (\sqrt{\boldsymbol{A}}u_{m'}(t), \sqrt{\boldsymbol{A}}w_j)\psi(t)\,\mathrm{d}t \to \int_0^T (\sqrt{\boldsymbol{A}}u(t), \sqrt{\boldsymbol{A}}w_j)\psi(t)\,\mathrm{d}t.$$

Therefore, we can pass to the limit in (28):

$$-\psi(0)(u_0, w_j) - \int_0^T (u(t), w_j)\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)\,\mathrm{d}t = \int_0^T [(\boldsymbol{G}(u(t)), w_j) - (\sqrt{\boldsymbol{A}}u(t), \sqrt{\boldsymbol{A}}w_j)]\psi(t)\,\mathrm{d}t. \tag{29}$$

Additionally, if $\psi \in \mathscr{D}(0,T)$, the following relation holds in the sense of $\mathscr{D}'(0,T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, w_j) = -(\sqrt{\boldsymbol{A}}u, \sqrt{\boldsymbol{A}}w_j) + (\boldsymbol{G}(u), w_j), \quad \forall j \in N. \tag{30}$$

Since $\sqrt{\boldsymbol{A}}u$ and $\boldsymbol{G}(u)$ are elements of $L^2(0,T;\boldsymbol{H})$, the scalar products on the right-hand side of (30) form regular distributions. Then $(\mathrm{d}/\mathrm{d}t)(u,w_j)$ is also a regular distribution. It means that relation (30) holds in $L^2(0,T)$.

Let us verify that the weak solution u satisfies the initial condition.

Multiplying Eq. (30) by a function $\psi \in C^1(0,T)$, for which $\psi(T)=0$, integrating through $(0,T)$, and integrating by parts on the left-hand side, the following equality is obtained:

$$-\psi(0)(u(0),w_j) - \int_0^T (u(t),w_j)\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)\,\mathrm{d}t$$
$$= \int_0^T [(\boldsymbol{G}(u(t)),w_j) - (\sqrt{\boldsymbol{A}}u(t),\sqrt{\boldsymbol{A}}w_j)]\psi(t)\,\mathrm{d}t. \tag{31}$$

Subtracting (31) from (29), we get

$$\psi(0)(u(0)-u_0,w_j)=0, \quad \forall j \in N \Rightarrow u(0)=u_0 \text{ in } \boldsymbol{H}. \tag{32}$$

This means that u is the weak solution of Eq. (1) with the initial condition u_0.

3.5. Uniqueness of the weak solution

Assume that there are two weak solutions u and v of Eq. (1) satisfying the initial condition $u(0)=v(0)=u_0$, then, following (30), for any $j \in N$ and for almost every $t \in (0,T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),w_j) + (\sqrt{\boldsymbol{A}}u(t),\sqrt{\boldsymbol{A}}w_j) = (\boldsymbol{G}(u(t)),w_j),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(v(t),w_j) + (\sqrt{\boldsymbol{A}}v(t),\sqrt{\boldsymbol{A}}w_j) = (\boldsymbol{G}(v(t)),w_j).$$

Subtract these two equations, and examine the function $w{:=}u-v$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t),w_j) + (\sqrt{\boldsymbol{A}}w(t),\sqrt{\boldsymbol{A}}w_j) = (\boldsymbol{G}(u(t))-\boldsymbol{G}(v(t)),w_j).$$

Obviously, $w(0)=u(0)-v(0)\equiv 0$.

The last equation multiplied by $(w(t),w_j)$ and summed for all $j \in N$ (this is possible – see [10, Chapter 3]) implies:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||w(t)||^2 + ||\sqrt{\boldsymbol{A}}w(t)||^2 = (\boldsymbol{G}(u(t))-\boldsymbol{G}(v(t)),w(t))$$
$$\leqslant \tfrac{1}{2}||w(t)||^2 + \tfrac{1}{2}||\boldsymbol{G}(u(t))-\boldsymbol{G}(v(t))||^2 \leqslant \tfrac{1}{2}||w(t)||^2 + \tfrac{1}{2}\mathscr{L}^2||w(t)||^2,$$

where the Young inequality (19) for $\varepsilon=1$ and the Lipschitz-continuity of $\boldsymbol{G}$ were used.

The Gronwall lemma leads to

$$||w(t)||^2 \leqslant \mathrm{e}^{(\mathscr{L}^2+1)t}\cdot||w(0)||^2 = 0 \Rightarrow w(t)\equiv 0, \quad \forall t \geqslant 0.$$

As the procedure was performed for all $T>0$, the function u is defined on $(0,+\infty)$, it has values in $\boldsymbol{V}$, and thanks to a priori estimate (26), $\{u_{m'}\}_{m'}$ converges weak-star in $L^\infty(0,+\infty;\boldsymbol{V})$ besides stronger convergence on $(0,T)$, $\forall T>0$. □

4. Application to a particular reaction–diffusion model

We demonstrate the use of the method on the reaction–diffusion system Brusselator (see [3]):

$$\frac{\partial x}{\partial t} = \frac{D_x}{L^2}\frac{\partial^2 x}{\partial z^2} + A - (B+1)x + x^2 y, \tag{33}$$

$$\frac{\partial y}{\partial t} = \frac{D_y}{L^2}\frac{\partial^2 y}{\partial z^2} + Bx - x^2 y, \tag{34}$$

where A, B, D_x, D_y, and L are positive constants, $x(t,z)$, and $y(t,z)$ are functions of time $t \in \langle 0, +\infty)$ and of one space variable $z \in \langle 0, 1\rangle$. The equations are completed by boundary conditions

$$x(t,0) = A, \qquad x(t,1) = A, \qquad y(t,0) = \frac{B}{A}, \qquad y(t,1) = \frac{B}{A}, \tag{35}$$

and initial conditions

$$x(0,z) = x_0(z), \qquad y(0,z) = y_0(z). \tag{36}$$

The model describes a fictitious reaction of two species in an inert medium. We convert problem (33)–(36) into one with homogeneous boundary conditions. Defining the transformation

$$X(t,z) = x(t,z) - A, \quad Y(t,z) = y(t,z) - \frac{B}{A}, \tag{37}$$

we obtain the system

$$\frac{\partial X}{\partial t} = \frac{D_x}{L^2}\frac{\partial^2 X}{\partial z^2} + [(B-1)X + A^2 Y] + \left[2AXY + \frac{B}{A}X^2\right] + X^2 Y, \tag{38}$$

$$\frac{\partial Y}{\partial t} = \frac{D_y}{L^2}\frac{\partial^2 Y}{\partial z^2} + [-BX - A^2 Y] + \left[-2AXY - \frac{B}{A}X^2\right] - X^2 Y. \tag{39}$$

endowed with the homogeneous boundary conditions and with the initial conditions in the form

$$u(0) = u_0 := \left(x_0 - A; y_0 - \frac{B}{A}\right)^{\mathrm{T}}. \tag{40}$$

Denoting

$$u(t) = \begin{pmatrix} X(t,\cdot) \\ Y(t,\cdot) \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} \frac{D_x}{L^2}, & 0 \\ 0, & \frac{D_y}{L^2} \end{pmatrix}, \quad \boldsymbol{F}(u) = \boldsymbol{C}u + \boldsymbol{B}(u) + \boldsymbol{T}(u),$$

$$\boldsymbol{C} = \begin{pmatrix} B-1, & A^2 \\ -B, & -A^2 \end{pmatrix}, \quad \boldsymbol{B}(u) = \begin{pmatrix} 2AXY + \frac{B}{A}X^2 \\ -2AXY - \frac{B}{A}X^2 \end{pmatrix}, \quad \boldsymbol{T}(u) = \begin{pmatrix} X^2 Y \\ -X^2 Y \end{pmatrix},$$

problem (38)–(40) can be written as

$$\frac{\partial u}{\partial t} = \boldsymbol{D}\Delta u + \boldsymbol{F}(u),$$

$$u|_{\partial\Omega} = 0,$$

$$u|_{t=0} = u_0. \tag{41}$$

The Fréchet derivative of the mapping $\boldsymbol{F}$ is

$$\nabla \boldsymbol{F}(u) = \boldsymbol{C} + \boldsymbol{L}(u) + \boldsymbol{Q}(u),$$

where

$$\boldsymbol{L}(u) = \begin{pmatrix} 2\frac{B}{A}X + 2AY, & 2AX \\ -2\frac{B}{A}X - 2AY, & -2AX \end{pmatrix}, \qquad \boldsymbol{Q}(u) = \begin{pmatrix} 2XY, & X^2 \\ -2XY, & -X^2 \end{pmatrix}.$$

In this case, the dimensions $n=1$, $d=2$, the domain $\Omega=(0,1)$, and the spaces $\boldsymbol{H}=L^2(0,1)\oplus L^2(0,1)$, and $\boldsymbol{V}=H_0^{(1)}(0,1)\oplus H_0^{(1)}(0,1)$. The mapping $\boldsymbol{F}$ is composed of polynomials, and is therefore locally Lipschitz-continuous. The existence of invariant regions has been proved in [2] for the following cases:

$\mathcal{O}$ = tetragon with vertices $[-1.9, -2.725]$, $[-1, 12.275]$, $[22.6, -2.475]$,

$[22.6, -2.725]$ for $A = 2$, $B = 5.45$, $D_x = D_y$,

$\mathcal{O}$ = rectangle with vertices $[-0.05, -0.15], [0.1, -0.15], [0.1, 0.1]$,

$[-0.05, 0.1]$ for $A = 0.5$, $B = 0.4$, D_x and D_y arbitrary.

For the purpose of the nonlinear Galerkin method, we use an orthonormal basis of the phase space $\boldsymbol{H}$:

$$\left\{ w_j^{(1)} = \begin{pmatrix} \Phi_j \\ 0 \end{pmatrix},\ w_j^{(2)} = \begin{pmatrix} 0 \\ \Phi_j \end{pmatrix} \right\}_{j=1}^{\infty}, \quad \text{where } \Phi_j(z) = \sqrt{2}\sin(\mathrm{j}\pi z).$$

The approximation u_m and the correction term z_m are

$$u_m(t) = \sum_{j=1}^{m} \alpha_j(t) w_j^{(1)} + \sum_{j=1}^{m} \beta_j(t) w_j^{(2)},$$

$$z_m(t) = \sum_{j=m+1}^{M} \alpha_j(t) w_j^{(1)} + \sum_{j=m+1}^{M} \beta_j(t) w_j^{(2)},$$

where the coefficients α_j, β_j are given by the following system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_j = (-\lambda_j^{(1)} + B - 1)\alpha_j + A^2\beta_j + \sum_{i=1}^{M}\sum_{k=1}^{M}\left[\frac{B}{A}\alpha_i\alpha_k + 2A\alpha_i\beta_k\right](\Phi_i\Phi_k, \Phi_j)_{L^2(\Omega)}$$

$$+ \sum_{i=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M}\alpha_i\alpha_k\beta_l(\Phi_i\Phi_k\Phi_l, \Phi_j)_{L^2(\Omega)} \quad \text{for } j = 1, \ldots, m, \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta_j = -(\lambda_j^{(1)} + 1)\alpha_j - \lambda_j^{(2)}\beta_j - \frac{\mathrm{d}}{\mathrm{d}t}\alpha_j \quad \text{for } j = 1, \ldots, m. \tag{43}$$

$$(-\lambda_j^{(1)} + B - 1 + A^2 c_j)\alpha_j + \sum_{i=1}^{m} \sum_{l=m+1}^{M} \left[2\frac{B}{A}\alpha_i + 2A\beta_i + 2Ac_l\alpha_i \right] (\Phi_i\Phi_l, \Phi_j)_{L^2(\Omega)}\alpha_l$$

$$+ \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{l=m+1}^{M} (2\alpha_i\beta_k + c_l\alpha_i\alpha_k)(\Phi_i\Phi_k\Phi_l, \Phi_j)_{L^2(\Omega)}\alpha_l \tag{44}$$

$$= -\sum_{i=1}^{m}\sum_{k=1}^{m} \left[\frac{B}{A}\alpha_i\alpha_k + 2A\alpha_i\beta_k \right] (\Phi_i\Phi_k, \Phi_j)_{L^2(\Omega)}$$

$$-\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} \alpha_i\alpha_k\beta_l(\Phi_i\Phi_k\Phi_l, \Phi_j)_{L^2(\Omega)} \quad \text{for } j = m+1, \ldots, M,$$

$$\beta_j = c_j\alpha_j \qquad \text{for } j = m+1, \ldots, M, \tag{45}$$

where $c_j = -(1 + \lambda_j^{(1)})/\lambda_j^{(2)}$ and for $i, j, k, l = 1, \ldots, M$:

$$\lambda_j^{(1)} = \frac{D_x}{L^2}(\mathrm{j}\pi)^2, \quad \lambda_j^{(2)} = \frac{D_y}{L^2}(\mathrm{j}\pi)^2,$$

$$(\Phi_i\Phi_k, \Phi_j)_{L^2(\Omega)} = \begin{cases} 0 & \text{for } (i+j+k) \text{ even,} \\ -\frac{\sqrt{2}}{\pi}\left(\frac{1}{i+j+k} - \frac{1}{i+j-k} - \frac{1}{i-j+k} - \frac{1}{j+k-i}\right) & \text{else,} \end{cases}$$

$$(\Phi_i\Phi_k\Phi_l, \Phi_j)_{L^2(\Omega)} = \tfrac{1}{2}(\mathrm{nul}(i+j-k-l, i-j+k-l, i-j-k+l)$$
$$- \mathrm{nul}(i+j+k-l, i+j-k+l, i-j+k+l, -i+j+k+l)),$$

where $\mathrm{nul}(i_1, \ldots, i_n) = |\{ j \in \hat{n} | i_j = 0\}|$ means the number of the zeros in the n-tuple $(i_1, \ldots, i_n)$.

We have used the Brusselator system to investigate the nonlinear Galerkin method regarding the accuracy and ability to save the CPU time. We present numerical computations performed for the following set-up of the system parameters: $A = 2$, $B = 5.45$, $D_x = 0.008$, $D_y = 0.004$.

Fig. 1 compares the usual Galerkin method with $M = 50$ and the nonlinear Galerkin method for $M = 50$ and the variety of values m for case $L = 1.25$ and $t \in \langle 0, 6000\rangle$, $x_0 = A + \sqrt{2}\sin \pi z + \sqrt{2}\sin 2\pi z + \sqrt{2}\sin 3\pi z$, $y_0 = B/A + \sqrt{2}\sin \pi z + \sqrt{2}\sin 2\pi z + \sqrt{2}\sin 3\pi z$. It can be seen that the nonlinear Galerkin method for $M = 50$, $m = 40$ saves nearly one half of CPU time required for the computation. Fig. 2 demonstrates the difference of numerical solutions of the usual and the nonlinear Galerkin methods with the above mentioned settings using the norm in the space $\boldsymbol{H}$.

Next, two figures show the complexity of the dynamics of the solution in the case when $L = 1.91$, $t \in \langle 2000, 6000\rangle$, $M = 30$, $m = 15$, $x_0 = A + \sqrt{2}\sin 2\pi z$, $y_0 = B/A + \sqrt{2}\sin 2\pi z$. Fig. 3 presents the time evolution of values of the solution for $z = 0.5$, and Fig. 4 is the Poincaré map using the hyperplane $x(t, 0.3) = 2$.

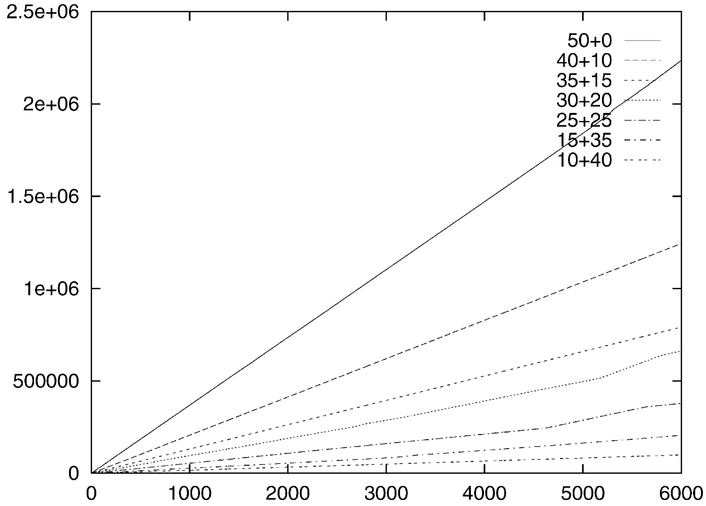

Fig. 1. The time-consumption comparison of the computations for $M=50$ and $m=50,40,35,30,25,15,10$, respectively, from above. The model time t is on the x-axis, the CPU time in seconds is on the y-axis.

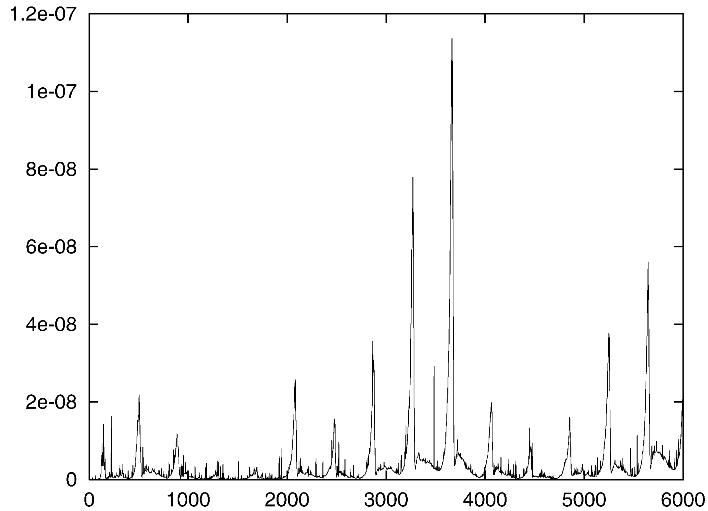

Fig. 2. The time-space comparison in terms of $||u_{2\times 40}-u_{2\times 50}||_H^2$ of the Galerkin method (dimension $M=50$) and the nonlinear Galerkin method ($m=40$, $M=50$). Time t is on x-axis, the norm of difference $u_{2\times 40}-u_{2\times 50}$ is on the y-axis.

5. Conclusion

The article presents a convergence analysis of the nonlinear Galerkin method applied to a system of reaction–diffusion equations admitting an invariant region. The method allows to approximate the solution for any finite time interval and saves a certain amount of CPU time. A generalization with respect to the dimensions of the approximating and correcting terms has been derived. The behaviour

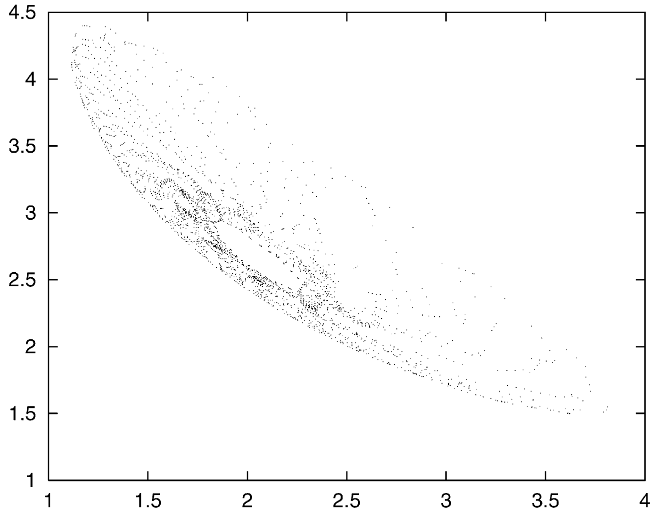

Fig. 3. The dynamics of Brusselator for $L = 1.91$. The graph contains time evolution of values $(x(t, \frac{1}{2}), y(t, \frac{1}{2}))$.

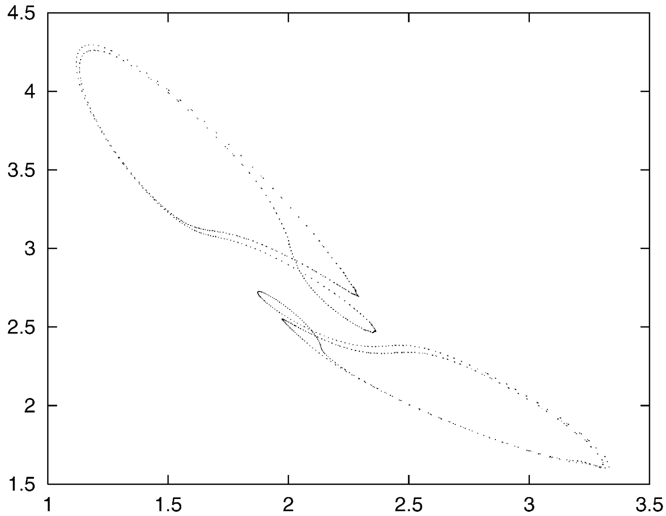

Fig. 4. The Poincaré map of Brusselator for $L = 1.91$. The graph contains values of $(x(t, \frac{1}{2}), y(t, \frac{1}{2}))$, when $x(t, 0.3) = 2$.

of the method has been demonstrated on the Brusselator reaction–diffusion scheme, where an optimal choice of the approximation and correction spaces allowed one half of the CPU time to be saved.

Acknowledgements

This work was partially supported by the project of the Czech Grant Agency under Contract No. 205/96/0921.

References

[1] A. Debussche, M. Marion, On the construction of families of approximate inertial manifolds, J. Differential Equations 100 (1992) 173–201.
[2] L. Eslerová, Numerical analysis of the dynamics of reaction–diffusion equations – in Czech, Diploma Thesis, Department of Mathematics FNSPE CTU, Prague, 1996.
[3] M. Holodniok, A. Klíč, M. Kubíček, M. Marek, Methods of the Analysis of Nonlinear Dynamical Models – in Czech, Academia, Prague, 1986.
[4] F. Jauberteau, C. Rosier, R. Temam, A nonlinear Galerkin method for the Navier-Stokes equations, Comput. Methods Appl. Mech. Eng. 80 (1990) 245–260.
[5] F. Jauberteau, C. Rosier, R. Temam, The nonlinear Galerkin method in computational fluid dynamics, Appl. Numer. Math. 6 (1989/90) 361–370.
[6] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod Gauthier-Villars, Paris, 1969.
[7] M. Marion, Attractors for reaction–diffusion equations: existence and estimate of their dimension, Appl. Anal. 25 (1987) 101–147.
[8] M. Marion, R. Temam, Nonlinear Galerkin methods, SIAM J. Numer. Anal. 26 (1989) 1139–1157.
[9] J. Smoller, Shock Waves and Reaction–Diffusion Equations, Springer, New York, 1983.
[10] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer, New York, 1988.